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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 64 NUMBER 3

CONTENTS

Flexagons . . . . . C. O. OAKLEY AND R. J. WISNER 143
A Derivation of the Equation for a Vibrating String F. A. FICKEN 155
The Exponential Function . . . . . JOHN G. KEMENY 158
Partial Linear Difference Equations . . . . TOMLINSON FORT 161
Semimagic Squares and Non-semisimple Algebras . ITIRO MURASE 168
The Solution of Linear Vector Differential Equations Containing Gyroscopic Terms . . . . . K. C. WESTFOLD 174
Mathematical Notes . H. KAUFMAN, F. KOEHLER, D. B. GOODNER, ... L. CARLITZ, JOHN LAMPERTI, RICHARD BELLMAN 181
Classroom Notes . . . LAURA GUGGENBUHL, A. W. MCGAUGHEY, . . F. B. HILDEBRAND, RAYMOND REDHEFFER, ROY DUBISCH 192
Elementary Problems and Solutions 197
Advanced Problems and Solutions 201
Recent Publications 205
News and Notices 207
The Mathematical Association of America 212
The Fortieth Annual Meeting of the Association 212
October Meeting of the Oklahoma Section 215
November Meeting of the New Jersey Section 217
November Meeting of the Northeastern Section 218
Officers and Committees as of January 1, 1957 220
Periods of Service of Former Officers of the Association . . . 222
By-Laws of the Mathematical Association of America (Inc.) . . 224
Calendar of Future Meetings 228

MARCH 1957

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THIS IS THE OFFICIAL JOURNAL OF THE

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(Devoted to the Interests of Collegiate Mathematics)

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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y., during the months of January, February, March, April, May, June-July, August-September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

FLEXAGONS

C. O. OAKLEY AND R. J. WISNER, Haverford College

1. Introduction. In 1939, four graduate students at Princeton University (R. Feynman, A. H. Stone, B. Tuckerman, and J. W. Tukey) discovered how to fold a piece of paper into what are now known as *flexagons*—hexagonal gadgets which "flex" under an operation we call pinching to exhibit several faces. Three short notes [3], [4], [6] merely show how to construct two of these paper models. So far as we know there is no other printed literature.

In order to motivate the definitions of Section 4, where abstract flexagons are discussed, we first describe informally how to construct physical models of the simplest abstract flexagons to be called regular flexagons of orders 3, 6, and 9.

2. Regular flexagons of orders 3 and 6. To construct the regular flexagon of order 3, RF_3, take a rectangular strip of paper* about an inch and a half wide and about a foot long, and from one end cut off a 30°, 60° triangle. Next, score

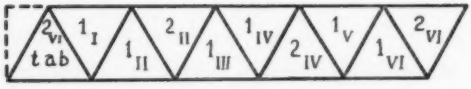

FIG. 1

the strip *very carefully* into ten equilateral triangles; discard any surplus material and label on both sides of the paper as in Figure 1. The first triangle on the left is a tab and is to be glued later to the last triangle on the right. With the strip oriented as in the figure, hold the tab in the left hand and with the right hand fold 2_{II} over on top of 1_{II}, fold 2_{IV} over on top of 1_{IV}, and fold 2_{VI} over on top of 1_{VI}. Now glue tab 2_{VI} onto 2_{VI} and the model is completed.

Hold the flexagon in the position of Figure 2 so that the Roman numerals I to VI, indicating what we shall call *pats*, run clockwise. Each of pats I, III, V contains a single triangle (piece of paper), each of pats II, IV, VI contains two triangles, and the pats are arranged in the form of a hexagon. Mark the vertices of the upper face of 2_{II} with the letters a, b, c clockwise as in Figure 2. Now with pat II to the north, pinch along the east radius (the east half of the east-west diagonal) forcing pats III and IV down. While holding these together push the west end of the west radius down. Actually this causes a folding east and along alternate radii and the flexagon begins to open at the center (now at the top). Releasing the pinching finger and thumb will permit the flexagon to "open" and lie flat again but this time displaying a new face (set of six triangles which will always appear together). The total operation is called a (physical) pinch. While

* Adding machine tape is satisfactory.

2_{II} is at the north, another pinch (east) is impossible since 1_{IV} and 2_{IV} are joined along the east radius. Rotate the whole model $-60°$ and label the vertices of 1_{II}, now north, with a, b, c as before. Pinch east. Rotate $-60°$ and label 1_I (north) with a, b, c. Another pinch brings the original face up again so that three distinct faces have appeared. If the flexagon is turned over, three pinches will exhibit three new (mathematical) faces since now the pat numbers run

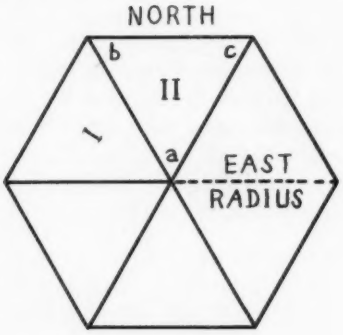

FIG. 2

counterclockwise. Moreover, the letter c is now at the center of the model.

The degree of a pat is the number of triangular pieces of paper in that pat. The order of a flexagon is the sum of the degrees of any two adjacent pats. Here the order is three.

To make the regular flexagon of order 6, RF_6, prepare a strip of nineteen triangles marked as in Figure 3. The construction is accomplished by "winding" the strip up pat by pat. In the folding of the triangles into pats, *the direction of the winding motion of the right hand is that of a wheel rolling on the ground toward you.* In what follows we suppress the Roman subscripts, which are the pat indicators, although they occur in the figure of the strip.

FIG. 3

To wind pat I:

(a) Hold strip as in Figure 3, tab in left hand;
(b) Fold triangle 2 back under triangle 1 by winding motion of right hand;
(c) Fold triangle 4 over on top of triangle 3 by winding motion;
(d) Fold the pair 4, 3 over on top of pair 1, 2 so that, from top down through

the pat, the strip numbers now read 3, 4, 1, 2. This completes the pat; it is of degree 4.

The order 3, 4, 1, 2 of this completed pat is unique; that is, there is no other way to assemble the four triangles in a 4-pat. For the moment we write this order 34,12 with a single comma placed where it tells us exactly what the final folding was in the winding process.

Pat II is a 2-pat and is wound as in pat II of RF_3 above. From top down through the pat the strip numbers are 2, 1 and there is no other way to assemble the two triangles into a 2-pat.

Now wind in order pat III the same as pat I, pat IV the same as pat II, pat V the same as pat I, and pat VI the same as pat II. The windings are therefore a triplication of pats I and II. Glue tab to 2_{VI} and the flexagon is complete. The pinching operation is the same for any flexagon. RF_3 and RF_6 are so special that we omit further discussion, but experimentation with them before proceeding would be helpful.

3. Regular flexagon of order 9. To make a model of the regular flexagon of order 9, RF_9, prepare a strip with a tab and 27 other triangles. Reserving the leftmost triangle for the tab, mark the others, from left to right, 1, 2, 3, 4; 1, 2, 3, 4, 5; · · ·, in triplicate. Wind pat I as in RF_6.

To wind pat II:

(a) Hold tab and pat I in left hand;
(b) Put 2 on top of 1;
(c) Put 4 under 3;
(d) Fold pair 3, 4 under pair 2, 1;
(e) Bring 5 over on top of 2 so that, from top down through the pat, the strip numbers read 5,2143. This completes the pat; it is of degree 5.

The position of the comma after 5 tells us that everything to the left of it was folded over 2143 in the last operation. The order 5,2143 is not unique for the 5-pat and this is explained in Section 5.

Triplicate these windings and glue tab. The strip numbers showing on the upper face should be 3's and 5's.

In order to keep track of all of the following pinches, ignore the original strip numbers and place a 1 in the middle of the top triangle of pat II; further, mark this triangle with *a*, *b*, *c* as in RF_3. There is no need of marking the other triangles since the same six pieces of paper will always appear together. This face is now designated as face 1*abc*. Record this face on another piece of paper as 1*a* using only the letter at the center. With pat II north, pinch east; label north 2*abc* recording this new face as 2*a*, and note that it is impossible to pinch east again. Rotate the flexagon $-60°$, pinch east, label new face (north) 3*abc* and record as 3*a*. Pinch, label north 4*abc*, record 4*a*. Pinch, label north 5*abc*, record 5*a*. Rotate, pinch; face 3*bca* appears but we record this as 3*b*, using only the letter at the center of the model.

Repeat the process: pinch east as many times as possible, then rotate and

pinch east, recording new faces as they arise. The succession of faces, from the beginning, is the following, the r indicating that a rotation is to be made before another pinch: $1a$, $2ar$, $3a$, $4a$, $5ar$, $3br$, $4a$, $2c$, $6ar$, $4cr$, $2c$, $3a$, $1a$, $7a$, $8ar$, $1cr$, $7a$, $3c$, $9ar$, $7cr$, $3c$; $1a$.

Remark 1. At any time a pinch following a rotation of $\pm 120°$ will yield the same face as the pinch without such a rotation. That is, pinching along any one of three alternate radii will open the flexagon to the same face. If the flexagon can also be pinched along one of the other set of alternate radii, a second face is exhibited.

Remark 2. In the middle of the sequence of pinches above, the original face $1a$ reappeared. But the next pinch led to the new face $7a$, and not to $2a$, because the flexagon was so oriented that one of the other set of alternate radii lay east. This situation is standard: in every complete sequence of pinches, a given face from which two pinches are possible will appear twice and in the proper orientations to yield the two possible faces. In any flexagon, the faces with no more than a single opening are those and only those where there is but a single paper triangle in alternate pats.

The following definitions apply to all flexagons. A *physical face* is that collection of the six uppermost triangles, one to a pat, regardless of their orientation. Each different orientation of these six triangles, with respect to each other, determines what is called a *mathematical face.*

The above RF_9 has the following properties:

(a) It has 9 physical faces;
(b) It is a Möbius band of 21 half-twists;
(c) It requires 9 rotations and 21 pinches to run through a complete cycle on one side;
(d) It has a total of 30 mathematical faces, 15 on either side. (The pat labels run counterclockwise for each back side mathematical face.)

4. Definition of an abstract flexagon. Take a new flexagon strip, mark with tab, 1, 2, 3, 4, 5 and wind a 5-pat as in RF_9. Now turn the whole strip over noting that the order 52143 has been reversed to read 34125. Mark the next two triangles in the strip 6 and 7, and with these wind a 2-pat which will of course read 76. If this 2-pat is now folded over onto the reversed 5-pat, the numbers 76 are reversed. The total process is that of winding a 7-pat reading, top to bottom, 67,34125. This helps to motivate the following definitions, which in this section are concerned with abstract flexagons. But we do not continue to carry the adjectives "abstract" and "physical" except where confusion might arise.

Let m be a positive integer. For $m=1$, the single permutation of the integer 1 is called a *pat of degree* 1. For $m=r+s>1$, the permutation $A_rA_{r-1}\cdots A_2A_1b_sb_{s-1}\cdots b_2b_1$, where $A_i=a_i+s$, of the integers 1 to m is called a *pat of degree* m if the permutation $a_1a_2\cdots a_r$ of the integers 1 to r and the permutation $b_1b_2\cdots b_s$ of the integers 1 to s are pats of degree r and s, respectively. We define an *abstract flexagon* F to be an ordered pair of pats, $F=(P, Q)$. If the pats

are of degree p (a p-pat) and q (a q-pat), then $N=p+q$ is called the *order* of the flexagon F_N.

Two important operations on F_N which preserve the order N are a pinch and a rotation. If the degree of Q is at least two, a *pinch* is that transformation which carries F into F', where $F=(A_rA_{r-1}\cdots A_1b_sb_{s-1}\cdots b_1,\ C_tC_{t-1}\cdots C_1d_ud_{u-1}\cdots d_1)$; $F'=(D_uD_{u-1}\cdots D_1b_1b_2\cdots b_sA_1A_2\cdots A_r,\ c_1c_2\cdots c_t)$; $A_i=a_i+s$, $C_i=c_i+u$, $D_i=d_i+r+s$, and it is easily seen that F' is a flexagon. A *rotation* of a flexagon is the transposition of its pats.

Two flexagons are said to be *equivalent* if one is obtainable from the other by a sequence of pinches or rotations, and this is an equivalence relation.

A flexagon can be represented physically as a triplication of a pair of pats, the arrangement in the paper model being such that the three pairs form a flexible hexagon which, under a pinch, exhibits another face. It can be constructed from a straight piece of paper by folding and gluing. For $N=1+1$, F_2 is the ordinary hexagon.

There is a subclass of flexagons, closed under the operations of pinching and rotating, which constitutes a universal class from which all flexagons arise. A member of this class is called a *regular flexagon* RF_N and is defined as above but in terms of *regular pats* which require $m\not\equiv 0$, mod 3. Further, the degrees p and q of the ordered pair of regular pats must belong to different residue classes, mod 3. Hence p, q are of the form $3k+1$, $3k+2$ and $N=p+q\equiv 0$, mod 3. A model of a regular flexagon can be constructed from a straight piece of paper by folding only.

The next two sections are devoted to regular flexagons.

5. Regular pats. We now consider the number of distinct ways to wind some regular pats of degree m ($\not\equiv 0$, mod 3). Pats of degree $m=1$ and $m=2$ can be constructed in only one way.* A pat of degree $m=4$ is also unique, namely 34,12.

If $m=5$, there are two and only two distinct ways. To make a 5-pat, we add one more triangle to the 4-pat. But clearly this can be done in two ways: first, to the whole 4-pat, *after it has been turned over*, can be added triangle 5, which is now on top. In turning the 4-pat over, we have reversed its sequential order so that the whole 5-pat now reads, top to bottom, 5,2143. Second, a 4-pat could be wound on triangles 2, 3, 4, 5 and its order (3412) would then be 4523. Now *this* can be turned over, becoming 3254, and folded over triangle 1 making the 5-pat read 3254,1. Note that the ordered binary partitions of $5=r+s$, where neither r nor s is congruent to 0, mod 3, are $1+4$ and $4+1$. These must be considered in winding the 5-pat which is necessarily made up of $1+4$ triangles or $4+1$ triangles. There are no other ways to wind a regular 5-pat.

If $m=7$, there are four ways. The permissible ordered binary partitions are $7=2+5$ and $7=5+2$. We take *one* of the 5-pats, say 52143 made from triangles

* We are not concerned here with $m\equiv 0$, mod 3. If you try to wind a 3-pat, for example, you will see that the paper folds back over the tab and the winding cannot proceed. (But see Section 7.)

1, 2, 3, 4, 5, and we take the only 2-pat now to be called 76 because it is made from triangles 6 and 7 in the strip, and combine them. Physically each of these must be turned over to be combined and therefore, sequentially, the 7-pat reads 67,34125. Using, in the same way, the 32541, we get another 7-pat, 67,14523. Or, for the binary partition $7=5+2$, we write the $5=1+4$ sequence, namely 52143, on the triangles 3, 4, 5, 6, 7, thus adding 2 to each member of the sequence which now becomes 74365. Combining this with 21 (reversing each since, physically, they have to be turned over) we arrive at the first sequence corresponding to the partition $7=5+2$, namely, 36745,12. Similarly for the other $7=5+2$, namely, 56347,12. There are no more ordered partitions of 7 and, consequently, no more 7-pats.

We have been doing nothing more than forming regular pats from the definition. Following is a table of regular pats extending through $m=8$. The notation of a pat describes how to wind it. Consider for example the last entry of Table I, and remember the *direction* of winding. Put 5 and 4 together (5 under 4); 7 over 6; 7 (along with 6) over 4; 3 under 2; 8 over 5; 8 (and everything under it) on 2; 6 (and everything under it) on 1.

TABLE I. REGULAR PATS OF DEGREE m

m	Pat	m	Pat
1	1	$8=1+7$	8,5214376
$2=1+1$	2,1		8,2154763
$4=2+2$	34,12		8,2174365
$5=1+4$	5,2143		8,3254176
$=4+1$	3254,1	$=4+4$	6587,2143
$7=2+5$	67,34125	$=7+1$	3265874,1
	67,14523		6325487,1
$=5+2$	36745,12		4365287,1
	56347,12		3285476,1

The *thumbhole* in any pat is that unique place such that each number to the right of it is less than each number to the left. Actually, the comma as it has been used indicates the thumbhole. Since the thumbhole is unique, the comma will now be omitted. The thumbhole separates a pat according to the partitioning used in the winding and is the one place in the physical pat where the thumb can be inserted without encountering a pocket.

Under a pinch east of any flexagon, with pat II north:

(a) Those triangles in I are retained in I but are reversed;

(b) Those triangles above (to the right of) the thumbhole in II are retained in II but are reversed;

(c) Those triangles below (to the left of) the thumbhole in II are slid out of II, without reversal, onto the top of I (reversed).

In the notation of the definition of a flexagon, our example RF_9 becomes (3412,52143). By (a), (b), and (c), the first pinch produces the ordered pair of

regular pats (65872143,1) where the structure of each pat is clearly exhibited. (Of course, we mentally relabel the triangles in writing new pat structures.) This "new pat I" is the one 8-pat derived from the partition $8=4+4$ (See Table I). This corresponds to face $2a$. We rotate, pinch, *etc.* The total sequence is as follows:

$1a$	3412,52143	$2c$	3674512,21		1,85214376
$2ar$	65872143,1	$6ar$	82154763,1	$1cr$	63254871,1
	1,65872143		1,82154763		1,63254871
$3a$	32541,3412	$4cr$	32658741,1	$7a$	21,6734125
$4a$	6714523,21		1,32658741	$3c$	5634712,21
$5ar$	83254176,1	$2c$	21,3674512	$9ar$	82174365,1
	1,83254176	$3a$	3412,32541		1,82174365
$3br$	43652871,1	$1a$	52143,3412	$7cr$	32854761,1
	1,43652871	$7a$	6734125,21		1,32854761
$4a$	21,6714523	$8ar$	85214376,1	$3c$	21,5634712

Each pinch and rotation has produced another flexagon. As a matter of fact, each pat in Table I has been used. This equivalence class of flexagons could have been made originally according to any entry in the above sequence. For example, look at $3a$ 3412,32541; this is an RF_9 wound with the *other* regular 5-pat.

6. Number of regular flexagons. To find the number of regular flexagons of a given order we must first determine the number u_{3k+1} and the number u_{3k+2} of distinct regular pats of degree $3k+1$ and $3k+2$. We have,

$$u_1 = 1, \tag{1}$$

$$u_{3k+1} = u_2u_{3k-1} + u_5u_{3k-4} + \cdots + u_{3k-1}u_2, \tag{2}$$

$$u_{3k+2} = u_1u_{3k+1} + u_4u_{3k-2} + \cdots + u_{3k+1}u_1, \tag{3}$$

where the subscripts describe the ordered partitions. We define the two generating functions $f(x) = \sum_{k=0}^{\infty} u_{3k+1}x^k$ and $g(x) = \sum_{k=0}^{\infty} u_{3k+2}x^k$ and it follows from (1), (2), and (3) that $g(x) = f^2(x)$, $f(x) = 1 + xg^2(x)$, so that

$$f(x) = 1 + xf^4(x), \tag{4}$$

$$g(x) = [1 + xg^2(x)]^2. \tag{5}$$

To compute the coefficients u_{3k+1} and u_{3k+2} we apply Lagrange's inversion formula [8] directly to (4) and after the transformation $h(t) = tg(t^2)$ to (5). The results* are

$$u_{3k+1} = \frac{1}{4k+1}\binom{4k+1}{k}, \qquad u_{3k+2} = \frac{1}{2k+1}\binom{4k+2}{k}.$$

* We are indebted to T. S. Motzkin and Hans Rademacher who, independently, transmitted them to us in correspondence.

The total number U_N of regular flexagons of order N, each being an ordered pair of regular pats, is therefore given by

$$U_N = U_{2\lambda} = 2(u_1u_{2\lambda-1} + u_4u_{2\lambda-4} + \cdots + u_{2\lambda-2}u_2)$$
$$= 2\sum_{\gamma=0}^{\lambda-1} \frac{1}{4\gamma+1}\binom{4\gamma+1}{\gamma} \cdot \frac{2}{4(\lambda-\gamma-1)+2}\binom{4(\lambda-\gamma-1)+2}{\lambda-\gamma-1}.$$

This can be summed by making use of a result of Gould [2], namely: $\sum_{k=0}^{n} A_k(a, b)A_{n-k}(c, b) = A_n(a+c, b)$, where

$$A_m(\alpha, \beta) = \frac{\alpha}{\alpha+\beta m}\binom{\alpha+\beta m}{m}.$$

Therefore,

$$U_{2\lambda} = 2\sum_{\gamma=0}^{\lambda-1} A_\gamma(1, 4)A_{\lambda-\gamma-1}(2, 4) \quad = \frac{6}{4\lambda-1}\binom{4\lambda-1}{\lambda-1}.$$

But these are divided into equivalence classes, and so in that sense the number of unique regular flexagons is considerably smaller than the number of ways in which they may be constructed. We shall now compute the number of equivalence classes.

Each (physical) triangle must—at some stage of pinching—constitute a pat by itself. For if in any pat P of degree exceeding one, we fix on any particular triangle T, then the flexagon can be held (turned over if necessary) so that T is above the thumbhole in P (north) and a pinch (east) reduces the degree of P which, of course, still contains T. Repetition of this process will reduce the pat P containing T to degree one. (Here, P is used as a generic notation for the new pats containing T; and the turning over of the flexagon causes no loss of generality.)

In each adjacent pair of pats there is a sum of N triangles and each time one of them constitutes a pat by itself, a rotation is necessary to continue the normal course of pinching. Hence, there are at most N rotations in the course of pinching through all the physical faces. Furthermore, for each pair of adjacent pats, each pinch annexes triangles to one of the pats by a half-twist and at the same time removes triangles from the other by means of removing a half-twist from that pat. Since a flexagon of order N is a Möbius band of $3N-6$ half-twists, there will be at most $3N-6$ pinches in running through a class of equivalent flexagons. Therefore, a flexagon runs through at most $4N-6$ stages by means of rotations and pinches. For RF_N it is easy to show that the number of stages is either $4N-6$, (full period), or $(4N-6)/3$, (1/3 period). Period 1/3 occurs when and only when the flexagon is equivalent to $(a_1a_2 \cdots a_m, A_mA_{m-1} \cdots A_1a_ma_{m-1} \cdots a_1)$. For the argument see Section 7.

The number of equivalence classes is therefore $U_N/(4N-6)$ in case the period

is full. But in cases where the period is 1/3 this number must be increased by $2u_m/3$. Where $N=3\lambda$, we now take cases.

(a) $\lambda=3k$. Since no regular pat has degree a multiple of three, no flexagon of period 1/3 is possible in this case and the number of equivalence classes is

$$U_N^* = \frac{6}{(12\lambda-6)(4\lambda-1)}\binom{4\lambda-1}{\lambda-1} = \frac{1}{(6k-1)(12k-1)}\binom{12k-1}{3k-1}.$$

(b) $\lambda=3k+1$. Here we must add $2u_{3k+1}/3$.

$$U_N^* = \frac{6}{(12\lambda-6)(4\lambda-1)}\binom{4\lambda-1}{\lambda-1} + \frac{2}{3(3k+1)}\binom{4k}{k}$$

$$= \frac{1}{3(4k+1)(6k+1)}\binom{12k+3}{3k} + \frac{2}{3(3k+1)}\binom{4k}{k}.$$

(c) $\lambda=3k+2$. Similar to (b), we obtain

$$U_N^* = \frac{1}{3(2k+1)(12k+7)}\binom{12k+7}{3k+1} + \frac{4}{3(3k+2)}\binom{4k+1}{k}.$$

7. General flexagons. We now wish to discuss non-regular pats and hence remove the restriction that the order of a flexagon be a multiple of three. Consider the regular pat 3412; if we identify the triangles numbered 3 and 4, we then have a pat of degree three which we may write as $\overline{34}12$, or simply 312. This topological identification of triangles may be executed physically by gluing triangles 3 and 4 together. It is easily seen that such identification may be made on any pair of consecutive integers which are adjacent numbers of the pat. Hence, by identifying 1 and 2 in 3412, we obtain $34\overline{12}$, or simply 231. These are the only possibilities for a general pat of degree three. To illustrate further, we obtain the general pats derivable from the regular 5-pat 52143. These are $5\overline{21}43 = 4132$, $521\overline{43} = 4213$, and $5\overline{2143} = 312$. All the non-regular pats may be obtained from the regular ones in this manner, and they may be combined without restriction (physically: by gluing when necessary) to form new pats of any degree and flexagons of any order. The definitions of a pat, a flexagon, a pinch, and a rotation are given in Section 4.

Pats of degree m are dependent on the ordered binary partitions of $m=r+s$. For example, two 7-pats are obtained from 4213 and 312 by first taking $a_1a_2a_3a_4 = 4213$, $b_1b_2b_3 = 312$ (yielding 6457213) and by next taking $a_1a_2a_3 = 312$, $b_1b_2b_3b_4 = 4213$ (yielding 6573124). The pats of degrees 1 through 5 are given by: **1**; **21**; 312, 231; 3241, 2431, **3412**, 4213, 4132; 25341, 24531, **32541**, 42351, 34251, 43512, 35412, 45213, 45132, 51423, 51342, **52143**, 53124, 52314. The regular pats are given in bold face type.

We now obtain the number of non-equivalent flexagons of order N. If v_m is

the number of pats of degree m, then $v_m = \sum_{k=1}^{m-1} v_{m-k}v_k$. This recursive convolution may be solved by consideration of the generating function $\phi = \sum_{k=1}^{\infty} v_k x^{k-1}$, which satisfies $x\phi^2 - \phi + 1 = 0$ and for which we impose the condition $v_1 = 1$ as demanded by the definition of a 1-pat. Again, by use of Lagrange's inversion formula, we find

$$v_m = \frac{1}{2m-1}\binom{2m-1}{m}.$$

The number V_N of flexagons of order N is the sum of the number of ways in which ordered pairs of pats may be taken, and hence

$$V_N = v_N = \frac{1}{2N-1}\binom{2N-1}{N}.$$

Notice that $V_N = \{(4N-6)/N\}\, V_{N-1}$.

As before, each flexagon may be pinched or rotated into $4N-6$ equivalent flexagons except that the flexagon $(a_1a_2 \cdots a_m,\ A_mA_{m-1} \cdots A_1a_ma_{m-1} \cdots a_1)$ has 1/3 period and the flexagon $(a_1a_2 \cdots a_m,\ a_1a_2 \cdots a_m)$ has 1/2 period. We must show that these are the only special cases to worry about in computing V_N^*, the number of non-equivalent flexagons of order N.

First, the two cases cited never coincide. That is, if $N = 6M$, then no flexagon of the form $(a_1a_2 \cdots a_{2M},\ A_{2M}A_{2M-1} \cdots A_1a_{2M}a_{2M-1} \cdots a_1)$ is equivalent to a flexagon of the form $(b_1b_2 \cdots b_{3M},\ b_1b_2 \cdots b_{3M})$. For this to occur, the $2M$-pat $a_1a_2 \cdots a_{2M}$ would have to be a duplication of an M-pat $P = c_1c_2 \cdots c_M$ (*i.e.*, $a_1a_2 \cdots a_{2M} = C_MC_{M-1} \cdots C_1c_Mc_{M-1} \cdots c_1$), and $b_1b_2 \cdots b_{3M}$ would have to be a triplication of the same pat P. Now if we identify the triangles of P as being the same triangle, then the flexagon $(a_1a_2 \cdots a_{2M},\ A_{2M}A_{2M-1} \cdots A_1a_{2M}a_{2M-1} \cdots a_1)$ becomes (21,3412) and $(b_1b_2 \cdots b_{3M},\ b_1b_2 \cdots b_{3M})$ becomes either (312,312) or (231,231). But (21,3412) is a regular flexagon, and hence is not equivalent to a non-regular one.

Second, let F_N be a flexagon of order N. We must establish that F_N is full period, 1/2 period, or 1/3 period. We consider the case in which F_N is a 1-pat and an $(N-1)$-pat, and no generality is lost since every flexagon is equivalent to one of this type. It is clear that if F_N is equivalent to fewer than $4N-6$ flexagons, it is necessarily equivalent to one which is composed of an m-pat P_m, $m>1$, and a k-plication of that same P_m. By identifying the triangles of P_m, we arrive at a derived flexagon which is a 1-pat and a k-pat. If $k=1$, then the derived flexagon is F_2, the hexagon, which has 1/2 period; F_N also has 1/2 period. If $k=2$, the derived flexagon is RF_3, of 1/3 period and F_N has 1/3 period. If $k>2$ and if the derived flexagon were full period, then F_N would also be full period. Since this is contrary to the hypothesis, the derived flexagon is not full period, and we repeat the process. Thus, we arrive at a flexagon which is either 1/2 period or 1/3 period, and F_N has the corresponding periodicity. Since RF_N is never of the form (P, P), a *regular* flexagon cannot be of 1/2 period.

Therefore, in computing V_N^*, we must add to V_N the quantity $\frac{1}{2}v_{N/2}V_2^*$ whenever $N\equiv 0$, mod 2, and $\frac{2}{3}v_{N/3}V_3^*$ whenever $N\equiv 0$, mod 3. But since V_2^* and V_3^* are both 1, we have

$$V_N^* = \frac{1}{4N-6}V_N + \left\{\frac{1}{2}\,v_{N/2}\right\} + \left\{\frac{2}{3}\,v_{N/3}\right\}$$

$$= \frac{1}{N}V_{N-1} + \left\{\frac{1}{2}\,v_{N/2}\right\} + \left\{\frac{2}{3}\,v_{N/3}\right\},$$

where the braces indicate inclusion of that term when and only when applicable.

We now give a table showing some values of the various numbers considered, and it is interesting to note how small the class of regular flexagons is in comparison with the class of all flexagons.

TABLE II

N	u_N	U_N	U_N^*	$V_N(=v_N)$	V_N^*
2	1	0	0	1	1
3	0	2	1	2	1
4	1	0	0	5	1
5	2	0	0	14	1
6	0	6	1	42	4
7	4	0	0	132	6
8	9	0	0	429	19
9	0	30	1	1,430	49
10	22	0	0	4,862	150
11	52	0	0	16,796	442
12	0	182	5	58,786	1,424
13	140	0	0	208,012	4,522
14	340	0	0	742,900	14,924
15	0	1,224	24	2,674,440	49,536
16	969	0	0	9,694,845	167,367
17	2394	0	0	35,357,670	570,285
18	0	8,778	133	129,644,790	1,965,058

8. Remarks. A. There is an essential difference between the class of all flexagons and the subclass of regular ones. The statement in Section 7 that $V_N=v_N$ gives the hint: the general flexagon $(a_1a_2\cdots a_r,\ b_1b_2\cdots b_s)$ could be studied and considered as just the pat $A_rA_{r-1}\cdots A_1b_sb_{s-1}\cdots b_1$. The regular flexagon, RF_N, $N\equiv 0$, mod 3, could not be studied by means of studying regular pats since no regular pats have degree $m\equiv 0$, mod 3.

B. Notice that if $P=a_1a_2\cdots a_m$ is a pat, so also is $P'=a_1'a_2'\cdots a_m'$, where $a_j'+a_{m-j+1}=m+1$. Thus, pats occur in *conjugate pairs* or are self-conjugate (*e.g.*, 3412). To construct P', one may label the triangles in the flexagon strip from right to left instead of from left to right, and then carry out the instructions for winding P. Suppose $F=(P, Q)$ is a model of a flexagon. We have discussed

pinching and rotating F, but another obvious operation is that of revolving F 180° about an axis through opposite vertices. This carries F into the *conjugate flexagon* (Q', P').

C. We have constructed a "one-sided" theory of flexagons, and this has led to a simple analysis by convolutions. In the theory of pats, the conjugate pats arose by considering *ordered* partitions. Hence, in computing V_N^*, we have counted twice those *models* arising from ordered binary partitions with unequal components. However, when the components of the partitions are equal, the models have been counted only once. In order to compute the number W_N^* of models of flexagons of order $N=2M$, we write $W_{2M}^* = (V_{2M}^* + v_M)/2$. There is no other corresponding change in Table II.

D. In much the same manner as one labels symmetry operations on the faces of a regular polyhedron to obtain the group of symmetries of that configuration, one can find groups associated with flexagons. For example, one group associated with F_3 is S_3. These usually turn out to be dihedral groups, and finding them is useful and interesting in the teaching of elementary group theory.

E. The number v_m of flexagons of order m adds to the very long list of combinatorial interpretations (ranging from election possibilities and postage stamps to continued fractions) of the recursion formula $v_1=1$; $v_m = \sum_{k=1}^{m-1} v_{m-k}v_k$ which has appeared—with variations—many times in the literature. Some interpretations are mentioned by Becker [1], and a paper by Motzkin [5] gives interpretations and generalizations. The numbers u_N of this paper constitute a variation, and the relevance of continued fractions is given by Touchard [7]. Other references are given in these papers.

Added in proof: An article by Martin Gardner on flexagons appeared in *Scientific American*, December, 1956. While non-mathematical in nature, the article indicates rather complete unpublished work by the inventors, and is an account of the interesting history of the gadgets.

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A DERIVATION OF THE EQUATION FOR A VIBRATING STRING

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Consider an *inextensible* string tautly stretched between two *fixed* points. If the italicized words are understood literally, then no displacement of the string is possible, for any displacement leaving the endpoints fixed would increase the length of some part of the "inextensible" string.

In many derivations of the differential equation for the vibrating string (the one-dimensional wave equation) it is not clear in what way, precisely, the literal meanings of the words "fixed" and "inextensible" are relaxed. We present here a discussion, similar in some respects to one in Osgood's *Advanced Calculus*, in which "fixed" is still understood literally, while "inextensible" is replaced by "slightly extensible" in a sense that will be clarified presently.

The string treated here is a continuous curvilinear distribution of matter of uniform (unstretched) density and uniform and constant elasticity. The string offers no resistance to bending; hence the only body force is the tension acting in the direction of the tangent. The string moves in such a way that no portion ever becomes slack and the elastic limit is never exceeded; the tension is therefore related to the extension by Hooke's law. All other forces are neglected.

The ends of the string are attached to two fixed points whose distance apart exceeds the unstretched length of the string, which may then rest in equilibrium along the line joining these points. This equilibrium position is taken to be horizontal, and the motion is assumed to take place in a vertical plane.

We use rectangular coordinates in the plane of the motion, with x-axis along the equilibrium line and origin at the left endpoint of the string; unit vectors along the axes are denoted by $\mathbf{i}$ and $\mathbf{j}$. We use the same symbol $\mathbf{R}$ both for a point and for its position vector with respect to the origin. Partial differentiation is indicated by subscript letters.

Let the point with equilibrium position $x\mathbf{i}$ be located, at time t, at $\mathbf{R}(x, t) = [x+u(x, t)]\mathbf{i}+v(x, t)\mathbf{j}$. We assume that $\mathbf{R}_{xx}$ and $\mathbf{R}_{tt}$ exist. At each instant the distance along the string from the origin to $\mathbf{R}(x, t)$ is given by

$$s(x) = \int_0^x [(1 + u_\xi)^2 + v_\xi^2]^{1/2} d\xi.$$

An explicit expression for the magnitude τ of the tension will now be obtained. Let a segment have unstretched length Δq, equilibrium length Δx, and displaced length Δs, and let ϵ denote the elasticity (Young's modulus, of dimensions force/elongation/length). In equilibrium, at tension τ_0, Hooke's law states that

$$\tau_0 = \epsilon(\Delta x - \Delta q)/\Delta q;$$

in fact, $x = q(1+\tau_0/\epsilon)$, where x and q are the equilibrium and unstretched lengths measured from the origin. In the displaced position $\tau = \epsilon(\Delta s - \Delta q)/\Delta q$. Eliminate

Δq and let $\Delta x \to 0$, getting $\tau = (\tau_0+\epsilon)s_x - \epsilon = \tau_0 + (\tau_0+\epsilon)(s_x-1)$.

Since the string is never slack, $\tau > 0$. Hence $s_x > \epsilon/(\tau_0+\epsilon) > 0$, s and x are smooth functions of each other, and $x_s s_x = 1$. (The displaced and equilibrium densities, ρ and ρ_0, are related by the equation $\rho s_x = \rho_0$.) The unit tangent $\mathbf{T}$ is given by

$$\mathbf{T} = [(1 + u_x)\mathbf{i} + v_x\mathbf{j}]x_s = \mathbf{R}_x/s_x,$$

and the tension at $\mathbf{R}$ is $\tau\mathbf{T}$.

The only force acting on a segment of equilibrium length Δx is the difference $\Delta(\tau\mathbf{T})$ between the tensions at the ends of the segment. By the mean value theorem, $\Delta(\tau\mathbf{T}) = (\tau\mathbf{T})_x \Delta x$, where the partial derivative is evaluated at an appropriate point inside the segment. Let the centroid of the displaced segment be denoted by $\overline{\mathbf{R}}$. Then Newton's law states that

$$(\rho_0 \Delta x \overline{\mathbf{R}}_t)_t = (\tau\mathbf{T})_x \Delta x.$$

Cancel Δx and then let $\Delta x \to 0$, getting the differential equation $\rho_0 \mathbf{R}_{tt} = (\tau\mathbf{T})_x$. A short calculation using this equation shows that the equation for angular acceleration is satisfied identically.

In order to obtain a simple explicit expression for $(\tau\mathbf{T})_x$, we impose three further assumptions.

From the expression for τ it is clear that the following assumptions (SE: slightly extensible) are equivalent:

SE 1 $$|\tau - \tau_0| \ll \tau_0;$$

SE 2 $$(\tau_0 + \epsilon)|s_x - 1| \ll \tau_0.$$

Here "$\ll$" means "can be neglected compared with." It follows in particular that $|s_x - 1| \ll 1$. Although SE restricts both the physical characteristics of the string and the kinematic characteristics of the motion, we state it verbally by saying merely that the string is slightly extensible.

Our second assumption, SI, is that the string is "slightly inclined" to the horizontal. More explicitly, if θ is the angle of inclination, we assume that:

SI $$\sin^2\theta \ll 1, \quad \text{or} \quad v_x^2 \ll s_x^2.$$

It follows that $1 - \cos^2\theta = 1 - (1+u_x)^2/s_x^2 \ll 1$, whence, *a fortiori*, $1 - |1+u_x|/s_x \ll 1$.

Finally, let us record explicitly the restrictions on the second derivatives actually used in order to arrive at the wave equation:

A (1) $$|v_x v_{xx}| \ll |u_{xx}| \quad \text{and} \quad (2) \qquad \epsilon|v_x u_{xx}| \ll \tau_0 |v_{xx}|.$$

No clear physical or geometric motivation for this assumption has appeared. It is a useful consequence of A(1), in the presence of SE and SI, that $s_{xx} = u_{xx}$ approximately, as is clear from the equation $s_x s_{xx} = (1 + u_x)u_{xx} + v_x v_{xx}$.

Returning to the differential equation, we write

$$\rho_0 \mathbf{R}_{tt} = (\tau \mathbf{T})_z = [(\tau_0 + \epsilon - \epsilon/s_z)\mathbf{R}_z]_z$$
$$= (\tau_0 + \epsilon - \epsilon/s_z)\mathbf{R}_{zz} + (\epsilon s_{zz}/s_z^2)\mathbf{R}_z,$$

and approximate s_{zz} by u_{zz}. The result for the longitudinal component is

$$\rho_0 u_{tt} = [\tau_0 + \epsilon - (\epsilon/s_z)(1 - (1 + u_z)/s_z)]u_{zz}.$$

By SE and SI the term involving the parentheses may be neglected in comparison with ϵ, leaving $\rho_0 u_{tt} = (\tau_0 + \epsilon)u_{zz}$.

The result for the transverse component is

$$\rho_0 v_{tt} = (\tau_0 + \epsilon - \epsilon/s_z)v_{zz} + (\epsilon v_z/s_z^2)u_{zz}.$$

By SE, $(s_z - 1)/s_z$ may be replaced by $s_z - 1$ and then, by SE(2), $\epsilon(s_z - 1)$ may be neglected in comparison with τ_0. Finally, by SE and A, the last term may be neglected in comparison with the first, leaving $\rho_0 v_{tt} = \tau_0 v_{zz}$.

Thus the desired equations have been obtained. One notes that the speeds of propagation are different.

It may be of some interest to verify, as would indeed be expected, that these equations are the Euler equations for the problem of minimizing the integral

$$\int_{t_0}^{t_1} (T - V)dt,$$

where T and V are the kinetic and potential energies of the string.

If a segment of the string has unstretched length a then the work done in stretching it from length $z > a$ to length $z + \Delta z$ is $\tau \Delta z = \{\epsilon(z - a)/a\}\Delta z$. Hence the work done in stretching it from length a to length b is

$$\int_a^b \{\epsilon(z - a)/a\}dz = (\epsilon/2a)(b - a)^2.$$

In our earlier notation, then, the energy in a segment of equilibrium length Δx is $(\epsilon/2\Delta q)(\Delta s - \Delta q)^2 = (\epsilon/2q_x)(s_x - q_x)^2 \Delta x = \{\tau^2/2(\tau_0 + \epsilon)\}\Delta x$.

Hence, if l denotes the equilibrium length of the string,

$$V = 1/\{2(\tau_0 + \epsilon)\} \int_0^l \tau^2 dx$$

and the integral to be minimized is

$$\frac{1}{2} \int_{t_0}^{t_1} \int_0^l [\rho_0(u_t^2 + v_t^2) - \tau^2/(\tau_0 + \epsilon)]dxdt.$$

The Euler equations are now easily seen to be precisely the components of the equation $(\rho_0 \mathbf{R}_t)_t = (\tau \mathbf{R}_z/s_z)_z$ obtained above.

THE EXPONENTIAL FUNCTION

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The purpose of this paper is to discuss a slightly unorthodox definition of the exponential function.† This definition allows us to derive the basic properties of e^x and $\log x$ by very elementary means. During the discussion I will make use of the following four results:

AI. If $a>b>0$, then $a^n>b^n$.

AII. If $1+x>0$, then $(1+x)^n \geqq 1+nx$.

AIII. $\lim_{n\to\infty} (1 + x/n^2)^n = 1$.

AIV. If $f(x)$ has a positive derivative for all x, then f has an inverse function g defined for all values of $f(x)$, and if $y=f(x)$ then $g'(y)=1/f'(x)$.

The first two results are simple exercises in mathematical induction. The third result can be established by bounding the expression by $1+x/n$ from below and by 1 or $1/(1-x/n)$ from above. Finally, AIV is a standard theorem in the calculus.

As motivation for the discussion I suggest a consideration of compound interest and radioactive decay. At simple interest, of rate x, a dollar yields $(1+x)$; if compounded twice, it yields $(1+x/2)^2$; and if we keep halving the time interval, we arrive at the formula $(1+x/2^n)^{2^n}$. If x is negative, then the same formula applies to decay. The function $E(x)$ defined below may be thought of as interest compounded continuously or as continuous decay.

Definitions.

(1) $$E_n(x) = (1 + x/2^n)^{2^n}.$$

(2) $$F_n(x) = 1/E_n(-x).$$

(3) $$E(x) = \lim_{n\to\infty} E_n(x).$$

(4) $$F(x) = \lim_{n\to\infty} F_n(x).$$

The first definition holds for all x and n. The second holds for any x for sufficiently high n, high enough to guarantee that $E_n(x)$ is positive. That the two limits exist will be shown below.

THEOREM I. *$E_n(x)$ is identically 1 if $x=0$; monotone increasing in n for other values of x, for sufficiently high values of n.*

This result can be read off from the identity $E_{n+1}(x) = (1+x/2^n+x^2/2^{2n+2})^{2^n}$.

* I wish to express my thanks to James K. Schiller, on whose notes covering a lecture of mine I have based this article.

† The definition was suggested by E. Artin.

For sufficiently high n, $1+x/2^n>0$, and $E_{n+1}(x)>E_n(x)$ by AI, the equality holding only for $x=0$. Hence all we need to show, to prove convergence, is that $E_n(x)$ is bounded from above.

THEOREM II. *$F_n(x)$ is identically 1 if $x=0$; monotone decreasing in n for other values of x, for sufficiently high values of n.*

For sufficiently high n we have that $E_{n+1}(x)>E_n(x)>0$. Hence $1/E_{n+1}(x)<1/E_n(x)$. This holds for all x, including $-x$, which yields the theorem.

THEOREM III. *$E_n(x)<F_n(x)$ for all sufficiently high n.*

This follows from the identity $E_n(x)/F_n(x)=(1-x^2/2^{2n})^{2^n}<1$.

THEOREM IV. *The limits $E(x)$ and $F(x)$ exist and are positive for all x.*

They are limits of monotone increasing (decreasing) sequences which are bounded from above (below), and which stay above 0 for sufficiently large n.

THEOREM V. $F(x)=E(x)$.

We computed $E_n(x)/F_n(x)$ above. By AIII this tends to 1. Hence the numerator and the denominator have the same limit.

THEOREM VI. $E(-x)=1/E(x)$, $E(0)=1$.

The first result follows from the fact that $E_n(-x)=1/F_n(x)$, and Theorem V. The value of $E(0)$ can be found from this, or from the fact that $E_n(0)\equiv 1$.

THEOREM VII. $1+x\leqq E(x)\leqq 1/(1-x)$, *the right side holding for $x<1$.*

The left side inequality is easily derived from the definition of $E_n(x)$ and AII. Then we can substitute $-x$ for x in the inequality, and take reciprocals to get the right side.

THEOREM VIII. $E(x)\cdot E(y)=E(x+y)$.

If we let $x+y+z=0$ and compute $E_n(x)\cdot E_n(y)\cdot E_n(z)$, we find that it is of the form $(1+k/2^{2n})^{2^n}$, and hence by AIII it tends to 1. This means that $E(x)\cdot E(y)=1/E(z)=E(-z)$. But $-z=x+y$.

THEOREM IX. $D_xE(x)=E(x)$.

From the previous theorem, $[E(x+h)-E(x)]/h=E(x)[E(h)-1]/h$. Hence we know that the derivative is proportional to $E(x)$; we need only find the proportionality constant, which is $\lim_{h\to 0}[E(h)-1]/h$. For small h, Theorem VII gives the bounds $(1+h-1)/h$ and $(1/(1-h)-1)/h$ for the quotient. Both bounds tend to 1 as h tends to 0.

THEOREM X. *$E(x)$ is continuous, monotone increasing, concave upward.*

Since a derivative exists everywhere, the function is continuous. Since $E(x)$ is positive and is both the first and second derivative of itself, the rest of the

theorem follows.

THEOREM XI. $E(x)^y = E(xy)$.

If y is an integer, then $E(x)^y$ is defined by repeated multiplication. The theorem follows from repeated use of Theorem VIII. If $y = p/q$, then $E(x)^y$ is the positive number which, when raised to the qth power, yields $E(x)^p$. It is easy to verify that $E(xp/q)$ has this property. Irrational powers are defined as limits of rational powers. Since the above equality holds for all rational y, we can infer its truth for all y from the continuity of the function E.

We have now found the most important properties of the function E. It is worth noting that we also have a method for getting good numerical approximations for $E(x)$. By Theorem VI we may restrict ourselves to positive x. Since $E(x) = E(x/k)^k$, we may restrict ourselves to very small x. For these, Theorem VII furnishes a very good approximation, namely $(1+x/k)^k$. This provides the connection with the customary definition.

From Theorem XI we see that $E(x) = E(1)^x$. Hence E is an exponential function. Let us call the base $E(1) = e$, then $E(x) = e^x$. (Among other possible definitions of e, we find that e is the yield of one dollar compounded continuously at 100% interest for a year.) We can now restate Theorems VI–XI in more traditional form.

From AIV we can infer that e^x has an inverse function defined for all positive values of x. (It is easy to see from Theorem VII that e^x takes on all positive values.) This function is $\log x$, the natural logarithm. The usual properties of the logarithm follow from the properties of e^x, together with the fact that $\log x$ is its inverse function. I will consider only two analytic properties.

THEOREM XII. $D_x \log x = 1/x$.

From AIV, $D_y \log y = 1/D_x e^x = 1/e^x = 1/y$. The theorem follows by putting x for y.

THEOREM XIII. *The function log is continuous, monotone increasing, and concave downward.*

Continuity follows from the existence of a derivative. Since the derivative $1/x$ is positive (for $x > 0$ where log is defined), the function is monotone increasing. Since the second derivative $-1/x^2$ is clearly negative, the function is concave downward.

The discussion should be completed by considering a^x and $\log_a x$ for any positive a. This is easily accomplished once we know that e^x takes on all positive values. Then $a^x = e^{(\log a)x}$, and all the properties of a^x and $\log_a x$ follow from the properties discussed above.

This approach to the exponential and logarithmic functions has appealed to me as one suitable for a bright freshman class, allowing me to prove all the important properties of these functions, without using up an undue amount of classroom time.

PARTIAL LINEAR DIFFERENCE EQUATIONS

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1. We shall consider a linear difference equation with two independent variables only, although much that is written can be immediately generalized. The independent variables are restricted to integral values and the coefficients of the dependent variable are defined at points of a rectangle called *defining points*. The rectangle will be referred to as the *defining rectangle*. The equation itself serves to connect the values of the dependent variable at certain neighboring points in the plane of the independent variables. It is frequently most easily described by a pattern. Thus an equation,

$$k_1(i,j)y(i-1,j) + k_2(i,j)y(i,j-1) + k_3(i,j)y(i+1,j) + k_4(i,j)y(i,j+1) + k_5(i,j)y(i,j) = R(i,j), \tag{1}$$

is readily described by means of the five point pattern

FIG. 1

Equation (1) will be referred to as a five point equation. Here, as usual, the points of the pattern will be said to cluster about (i, j). We may simply refer to the pattern of the equation or the pattern about (i, j). The terms linear, homogeneous, *etc.*, have the usual import.

DEFINITION 1. *The* order *of a linear difference equation is the maximum number of steps required to get from one point of the pattern to another where, moreover, the coefficients of the dependent variable in the outlying points of the outlying rows and columns of the pattern are never zero.*

Thus equation (1) is of the second order provided that neither k_1, k_2, k_3, nor k_4 is ever zero. The equation,

$$k_1(i,j)y(i,j) + k_2(i,j)y(i+1,j) + k_3(i,j)y(i+2,j) + k_4(i,j)y(i,j+1) + k_5(i,j)y(i+1,j+1) + k_6(i,j)y(i+2,j+1) + k_7(i,j)y(i,j+2) + k_8(i,j)y(i+1,j+2) + k_9(i,j)y(i+2,j+2) = 0, \tag{2}$$

is described by a nine point pattern as illustrated:

FIG. 2

Equation (2) is of the fourth order provided neither k_1 nor k_3 nor k_7 nor k_9 is ever zero.

The requirement of the nonvanishing of coefficients of outlying terms seems somewhat artificial at first and is, in fact, at times more than is necessary. However, it is to be recognized that in this work there is nothing corresponding to a singular point of a differential equation. The vanishing of a leading coefficient simply destroys the equation at the point in question.

It is not found advisable to describe the equation that we are considering more precisely than to say that it is linear of fixed patterns. We shall denote it by

$$L(y) = R(i, j). \tag{3}$$

DEFINITION 2. *We shall designate by the term* basic points *those points which fall within the pattern when* (i, j) *is any defining point.*

For equation (1) the basic points constitute a rectangle without the four corner points. It is pictured in Figure 3 for a five by seven defining rectangle. Note that the basic rectangle is seven by nine except for corner points.

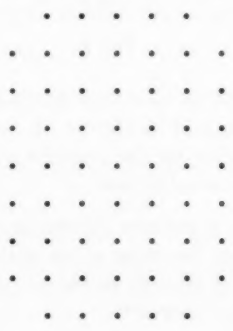

FIG. 3

On the other hand the basic points for an equation described by the nine point pattern of Figure 2 constitute all points of a rectangle.

We shall universally mean by the term *solution* a function that satisfies the equation at all points of a defining rectangle. However, it should be noticed that the solution is determined at more points than the coefficients are defined at,

namely the basic points. The reader will also notice that much that is said applies to other regions than rectangles.

DEFINITION 3. *A* fundamental domain *for a particular equation consists of points such that there exists one and only one solution taking on arbitrary prescribed values at each point of the domain.*

A fundamental domain for equation (1) with defining rectangle determined by $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$ consists of the points $(i, 0)$, $(i, 1)$, $i=1, 2, \cdots, m$ and $(0, j)$, $(m+1, j)$, $j=1, 2, \cdots, n$. These points can be described as the two bottom rows of the basic rectangle plus the first and last columns. There are other fundamental domains for this equation. To prove that the points listed constitute a fundamental domain simply mark them on the basic rectangle and notice that values for y at all points can be successively determined. We, of course, are assuming that k_1, k_2, k_3, k_4 are never zero. As a second illustration consider the nine point equation given by (2) with the same defining points. We readily show that the points of the two bottom rows and first two columns of the basic rectangle form a fundamental domain.

THEOREM 1. *If an equation involves a point pattern containing points from p consecutive columns and q consecutive rows then the points of $p-1$ columns and $q-1$ rows of the basic rectangle chosen as now to be described form a fundamental domain.*

Let the points in the top row of the pattern farthest to the right be $(i+s\ j+q-1)$ assuming (i, j) in the lower left corner. Then the following points form a fundamental domain; $q-1$ rows at the bottom of the basic rectangle; s columns at the extreme left and $p-s-1$ columns at the extreme right. Proof is by constructing the basic rectangle, marking the points just described and then filling out the basic rectangle point by point by means of the equation. There are other fundamental domains.

2. The homogeneous equation. Terms in common use with reference to differential or other types of equations will be used. It will not always be thought necessary to take the space to define them. The same will hold for simple theorems common to other homogeneous equations.

Suppose a certain fundamental domain for a homogeneous equation to consist of t points.

DEFINITION 4. *A* fundamental system *of solutions consists of t solutions, $y_1, \cdots, y_t$ which are such that when taken over the t points of some fundamental domain they constitute t sets of t linearly independent constants.*

For this it is, of course necessary and sufficient that the determinant of the t sets of the constants be different from zero.

THEOREM 2. *Any solution is a linear combination with constant coefficients of a fundamental system of solution.*

It is immediate that any solution is expressible linearly in terms of a fundamental system over a fundamental domain pertaining to that fundamental system. However, the determination of all values from the values over the fundamental domain is, by hypothesis, possible. This determination is from a set of linear equations and can be done by Cramer's rule. This will preserve the linearity.

THEOREM 3. *All fundamental systems contain the same number of solutions.*

Suppose $y_1, y_2, \cdots, y_t$ to be a fundamental system of solutions and $y_1', y_2', \cdots, y_{t'}'$ to be a second set. Suppose $t'>t$. We know that

$$y_1' = c_{11}y_1 + c_{12}y_2 + \cdots + c_{1t}y_t,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_{t'}' = c_{t'1}y_1 + c_{t'2}y_2 + \cdots + c_{t't}y_t.$$

This is true throughout the basic rectangle and hence in particular on a fundamental domain of $y_1', \cdots, y_{t'}'$. But $t'>t$. Hence $y_1', \cdots, y_{t'}'$ are linearly dependent over a fundamental domain of theirs. This contradicts our definition of a fundamental system. Hence $t' \leqq t$. Similarly $t \leqq t'$.

THEOREM 4. *All fundamental domains for a given equation contain the same number of points.*

This follows from the previous theorem since the number of solutions in a fundamental system is the same as the number of points in its fundamental domain.

THEOREM 5. *Let $y_1, y_2, \cdots, y_t$ be a fundamental system of solutions and let $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$ be a fundamental domain for them. Let $y_1', y_2', \cdots, y_t'$ be a second fundamental system. Then $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$ are a fundamental domain for $y_1', y_2', \cdots, y_t'$.*

If we assume the contrary to this theorem, then $y_1', y_2', \cdots, y_t'$ are linearly dependent over $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$. But $y_1 = d_{11}y_1' + d_{12}y_2' + \cdots + d_{1t}y_t', \cdots, y_t = d_{t1}y_1' + d_{t2}y_2' + \cdots + d_{tt}y_t'$. Hence $y_1, \cdots, y_t$ are linearly dependent over $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$. This contradicts our hypothesis.

DEFINITION 5. *Let $y_1, \cdots, y_t$ be solutions of an equation and let a fundamental domain for the equation contain t points. Let $(i_1, j_1), \cdots, (i_t, j_t)$ be any t points. Then the t by t determinant,* $\det(y_m(i_n, j_n))$ *is called the* Wronskian *of $y_1, \cdots, y_t$. We denote it by $W(y_1, \cdots, y_t)$.*

THEOREM 6. *A necessary and sufficient condition that $y_1, \cdots, y_t$ constitute a fundamental system is that $W(y_1, \cdots, y_t)$ does not vanish on any fundamental domain.*

Proof is immediate.

DEFINITION 6. *A formula which satisfies a difference equation and which has every solution of the difference equation as a special case is called the general solution.*

THEOREM 7. *If* $y_1, \cdots, y_t$ *constitute a fundamental system of solutions then* $c_1y_1+\cdots+c_ty_t$, *where* $c_1, \cdots, c_t$ *are arbitrary constants is the general solution.*

3. The non-homogeneous equation.

THEOREM 8. *The general solution of a non-homogeneous equation consists of a particular solution plus the general solution of the reduced (homogeneous) equation.*

4. Auxiliary conditions.

Write equation (3) for each defining point (i, j). Suppose that there are mn of them. We next assume that the basic rectangle contains $mn+\tau$ points. What we have is mn linear equations in $mn+\tau$ variables. We expect to be able, in some instances at least, to impose τ additional linear conditions and have all $mn+\tau$ equations satisfied. The most usual such conditions are boundary conditions.

We let the auxiliary condition be represented by

$$\bar{V}_1(y) = g_1, \cdots, \bar{V}_\tau(y) = g_\tau. \tag{4}$$

Here $g_1, \cdots, g_\tau$ are constants not all zero. Equations (4) together with (3) will be referred to as the *non-homogeneous* system. We shall call by the name *homogeneous* the system consisting of

$$L(y) = 0, \bar{V}_1(y) = 0, \cdots, \bar{V}_\tau(y) = 0. \tag{5}$$

It is, of course, assumed that $L, \bar{V}_1, \cdots, \bar{V}_\tau$ are linear and homogeneous.

Let $L(y)=0$ be written down for each of the defining points. We denote these equations by $L_{ij}(y)=0$. Now

$$L_{ij}(y) = 0, \bar{V}_1(y) = 0, \cdots, \bar{V}_\tau(y) = 0 \tag{6}$$

constitute $mn+\tau$ linear homogeneous equations in $y(i, j)$.

THEOREM 9. *In order for the equations* (5) *to be satisfied by a solution not identically zero it is necessary and sufficient that the determinant of the coefficients of* (6) *vanish.*

DEFINITION 7. *The determinant of the coefficients of* (6) *is called the* characteristic determinant *for system* (5).

DEFINITION 8. *If the characteristic determinant of system* (5) *vanishes then system* (5) *is said to be* compatible. *It is said to be* incompatible *in the contrary case.*

DEFINITION 9. *System* (5) *is said to be* K-fold compatible *in case the matrix of the coefficients is of rank* $mn+1-K$.

THEOREM 10. *If* (5) *is K-fold compatible there exist K, and not more, linearly independent solutions of system* (5).

Proof is immediate. There may, of course, be more than one set of linearly independent solutions.

We shall now assume (as is certainly many times the case) that the number of auxiliary conditions is the same as the number of points in a fundamental domain ($\tau = t$).

THEOREM 11. *Let $y_1, \cdots, y_t$ be a fundamental system of solutions of $L(y) = 0$. Then a necessary and sufficient condition that system* (5) *be compatible is that*

$$\det (V_m(y_n)) = 0. \tag{7}$$

Substitute $c_1y_1 + c_2y_2 + \cdots + c_ty_t$ in the auxiliary conditions. There result t equations for the determination of $c_1, c_2, \cdots, c_t$. The determinant of the coefficients of the c's is precisely the determinant of the theorem.

Consider next the non-homogeneous system

$$L(y) = R(i, j), \bar{V}_1(y) = g_1, \bar{V}_2(y) = g_2, \cdots, \bar{V}_t(y) = g_t. \tag{8}$$

THEOREM 12. *A necessary and sufficient condition that the system* (8) *have one and only one solution is that the* reduced system (5) *be incompatible.*

Proof follows immediately from Theorem 9.

5. Green's function. Consider the homogeneous system (5).

DEFINITION 10. *By the Green's function for* (5) *we shall mean a function $G(i, j; \xi, \eta)$, which as a function of (i, j) satisfies the auxiliary conditions and also satisfies the difference equation at every point of the defining rectangle except the point (ξ, η), and where, moreover, $L(G(\xi, \eta: \xi, \eta)) = 1$.*

THEOREM 13. *A necessary and sufficient condition that a Green's function $G(i, j; \xi, \eta)$ exist for every point of the defining rectangle is that* (5) *be incompatible.*

This theorem is again an immediate consequence of Theorem 9.

6. Construction of the Green's function for the five-point equation. We shall consider in some detail the construction of the Green's function for equation (1) subject to auxiliary conditions. The methods employed will be applicable to many other equations.

The defining rectangle is the rectangle determined by $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$. These points plus the boundary points $(0, j)$, $(m+1, j)$, $j = 1, \cdots, n$ and $(i, 0)(i, n+1)$, $i = 1, \cdots, m$ form what we have called the basic rectangle. Fundamental domains consists of certain $2m+2n$ points. Hence $t = 2m+2n$.

The auxiliary conditions are $\bar{V}_1(y) = 0, \cdots, \bar{V}_t(y) = 0$.

Let $y_1, \cdots, y_t$ be a fundamental system of solutions. Let $u_1 = c_1y_1 + c_2y_2 + \cdots + c_ty_t$ and $u_2 = d_1y_1 + d_2y_2 + \cdots + d_ty_t$. Assume $G(i, j; \xi, \eta) = u_1$ when $j \leqq \eta$ and $G(i, j; \xi, \eta) = u_2$ when $j > \eta$. We attempt to determine the c's and the d's so that $u_1 = u_2$ at all points where $j = \eta$ and where $j = \eta + 1$ except at the point

$(\xi, \eta+1)$ where $u_1-u_2=1/k_4(\xi, \eta)$ also, so that $u_1=u_2$ at all points of the first and last columns of the basic rectangle, that is, at the points $(0, j)$ and $(m+1, j)$, $j=1, \cdots, n$. If this is done $G(i, j; \xi, \eta)$ will satisfy the difference equation at all points of the defining rectangle except (ξ, η). At this point when G is substituted in the equation the result 1 is obtained as desired. To show that such a determination of the c's and d's is possible, let $z_v=c_v-d_v$, $v=1, 2, \cdots, t$. Then substitute u_1 and u_2 in the conditions just enumerated and transpose. We have $z_1y_1(i, j)+\cdots+z_ty_t(i, j)=0$, where (i, j) runs through all points on the lines $j=\eta$, $j=\eta+1$ except $(\xi, \eta+1)$ and also at all points on the lines $i=0$ and $i=m+1$. Moreover, $z_1y_1(\xi, \eta+1)+z_2y_2(\xi, \eta+1)+\cdots+z_ty_t(\xi, \eta+1)=1/k_4(\xi, \eta)$. The points which we have just described constitute a fundamental domain. We consequently can solve for $z_1, z_2, \cdots, z_t$. Now as to auxiliary conditions! Assign any values to the d's. Substitute in the auxiliary conditions and transpose all c's to the left-hand member and constant terms to the right-hand member. Solution for the c's is now possible since the determinant of the coefficients of the c's is precisely the determinant in (7) which is not zero, since the system is incompatible.

7. An application of Green's function. Consider the system consisting of

$$L(y) = R(i, j), \bar{V}_1(y) = 0, \bar{V}_2(y) = 0, \cdots, \bar{V}_t(y) = 0. \tag{9}$$

We speak of this as the *semihomogeneous* system.

THEOREM. *If the system* (5) *is incompatible then the system* (9) *has one and only one solution which is given by*

$$y(i, j) = \sum_{\eta=1}^{n} \sum_{\xi=1}^{m} G(i, j; \xi, \eta)R(\xi, \eta).$$

Proof is a matter of direct verification:

$$L(y(i, j)) = L\left[\sum_{\eta=1}^{n} \sum_{\xi=1}^{m} G(i, j; \xi, \eta)R(\xi, \eta)\right]$$

$$= \sum_{\eta=1}^{n} \sum_{\xi=1}^{m} L[G(i, j; \xi, \eta)]R(\xi, \eta) = R(i, j).$$

This follows since $L[G(i, j; \xi, \eta)]=0$ at all points except where $\xi=i$, $\eta=j$. Here it is 1 by definition of the Green's function. The satisfaction of the auxiliary conditions is immediate. The fact that there is no other solution follows from Theorem 9.

SEMIMAGIC SQUARES AND NON-SEMISIMPLE ALGEBRAS

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1. Recently, L. M. Weiner [1] wrote an interesting note about the so-called semimagic squares, which are defined as follows:

DEFINITION. *A semimagic square* $A=(a_{ij})$ *of order* n *is an* n *by* n *matrix for which* $\sum_{i=1}^{n} a_{ij}=\sum_{i=1}^{n} a_{ji}=\sigma(A)$ *for every* j.

The set of all semimagic squares of order n forms a subalgebra $\mathfrak{A}_n$ of the total matric algebra $\mathfrak{M}_n$ of degree n which satisfies the following conditions:

(a) $$\sigma(A+B)=\sigma(A)+\sigma(B),$$

(b) $$\sigma(kA)=k\sigma(A),$$

(c) $$\sigma(AB)=\sigma(A)\sigma(B).$$

He noticed a decomposition of $\mathfrak{A}_n$ into the direct sum of two ideals $\mathfrak{B}_n$ and $\mathfrak{C}_n$. The subalgebra $\mathfrak{B}_n$ consists of all matrices whose elements a_{ij} are all equal, and the subalgebra $\mathfrak{C}_n$, of all matrices whose $\sigma(A)$ is zero. Thus,

(1) $$A=\begin{pmatrix} \sigma/n & \cdots & \sigma/n \\ \cdots & \cdots & \cdots \\ \sigma/n & \cdots & \sigma/n \end{pmatrix}+\begin{pmatrix} a_{11}-\sigma/n & \cdots & a_{1n}-\sigma/n \\ \cdots & \cdots & \cdots \\ a_{n1}-\sigma/n & \cdots & a_{nn}-\sigma/n \end{pmatrix}.$$

Now, define A_{ij} to be the matrix with 1 in the first row and column, -1 in the first row and jth column, -1 in the ith row and first column, 1 in the ith row and jth column, and 0 elsewhere for $i, j=2, \cdots, n$. Then, as he remarked, the algebra $\mathfrak{C}_n$ has the dimension $(n-1)^2$, and the matrices A_{ij} form a basis for $\mathfrak{C}_n$.

These results are all he mentioned in his paper. But, in the following, it will be shown that $\mathfrak{C}_n$ is isomorphic to the total matric algebra of degree $n-1$ when the characteristic p of the field Ω of elements of matrices does not divide the order n, and that when n is divisible by p, the decomposition (1) does not hold, $\mathfrak{A}_n$ is not semisimple, and there is the radical $\mathfrak{N}$ such that $\mathfrak{N}^3=0$. Moreover, by taking half of the condition, either $\sum_{i=1}^{n} a_{ji}=\sigma(A)$ or $\sum_{i=1}^{n} a_{ij}=\sigma(A)$, for all j, we may obtain larger subalgebras of $\mathfrak{M}_n$, because the conditions (a), (b), (c) also hold. These algebras will be shown to be non-semisimple, having the radical $\mathfrak{N}$ such that $\mathfrak{N}^2=0$. The investigation of these algebras offers us a suitable illustration of the general theory of non-semisimple algebras [2].

2. We start from the half of the condition for semimagic squares.

DEFINITION. *A row-semimagic square* $A=(a_{ij})$ *of order* n *is a matrix of degree* n *for which* $\sum_{j=1}^{n} a_{ij}=\sigma(A)$ *for every* i.

The product C of two row-semimagic squares A and B is also row-semimagic, because

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} a_{ik}\sigma(B) = \sigma(A)\sigma(B).$$

Thus, the set of all row-semimagic squares of order n forms an algebra, which we denote by $\mathfrak{A}_n$, but, as we shall consider always a fixed order n, we may omit the subscript and simply write $\mathfrak{A}$.

Now consider the matrix $P=(p_{ij})$, where $p_{ii}=p_{in}=1$ $(i=1, \cdots, n)$ and $p_{ij}=0$ otherwise. When we transform a row-semimagic matrix A by P, then $A'=P^{-1}AP$ is given by

$$A' = \begin{pmatrix} a_{11}-a_{n1} & \cdots & a_{1,n-1}-a_{n,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a_{n-1,1}-a_{n1} & \cdots & a_{n-1,n-1}-a_{n,n-1} & 0 \\ a_{n1} & \cdots & a_{n,n-1} & \sigma \end{pmatrix}. \tag{2}$$

The set $\mathfrak{A}'$ of all $A'=(a'_{ij})$ forms an algebra isomorphic to $\mathfrak{A}$ and $\mathfrak{A}'$ consists of all matrices whose elements in the last column are all 0 except for a'_{nn}. Therefore, the dimension of $\mathfrak{A}$ is $n^2-(n-1)$. Now, we have

$$A' = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sigma \end{pmatrix} + \begin{pmatrix} a'_{11} & \cdots & a'_{1,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & \cdots & a'_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}. \tag{3}$$

The sum of the set of all matrices of the first and second types forms a subalgebra $\mathfrak{B}'$ of $\mathfrak{A}'$. The set $\mathfrak{N}'$ of all matrices of the third type is an ideal of $\mathfrak{A}'$, whereas $\mathfrak{B}'$ is not an ideal. Moreover, $\mathfrak{N}'$ is nilpotent and $\mathfrak{N}'^2=0$. The residue-class ring $\mathfrak{A}'-\mathfrak{N}'$ is isomorphic to $\mathfrak{B}'$, and is semisimple because $\mathfrak{B}'$ is the direct sum of a one-dimensional subalgebra and a subalgebra isomorphic to the total matric algebra of degree $n-1$. Hence, $\mathfrak{N}'$ is the radical of $\mathfrak{A}'$, and $\mathfrak{A}'=\mathfrak{B}'+\mathfrak{N}'$ is the supplementary sum of $\mathfrak{B}'$ and $\mathfrak{N}'$.

We return now to the algebra $\mathfrak{A}$. Here $\mathfrak{N}=P\mathfrak{N}'P^{-1}$ is the set of all matrices for which $\sigma(A)=0$ and $a_{1j}=\cdots=a_{nj}$ for $j=1, \cdots, n-1$, and consequently for $j=n$. Thus $\mathfrak{N}$ consists of all matrices of the type

$$(c_{ij}), \text{ where } c_{ij}=c_j\ (i,j=1, \cdots, n) \text{ and } \sum_{j=1}^{n} c_j = 0. \tag{4}$$

Corresponding to (3) we have the following decomposition of $\mathfrak{A}$:

$$A=\begin{pmatrix}0 & \cdots & 0 & \sigma\\ \cdot & \cdots & \cdot & \cdot\\ 0 & \cdots & 0 & \sigma\\ 0 & \cdots & 0 & \sigma\end{pmatrix}+\begin{pmatrix}a_{11}-a_{n1} & \cdots & a_{1n}-a_{nn}\\ \cdot & \cdots & \cdot\\ a_{n-1,1}-a_{n1} & \cdots & a_{n-1,n}-a_{nn}\\ 0 & \cdots & 0\end{pmatrix}+\begin{pmatrix}a_{n1} & \cdots & a_{n,n-1} & a_{nn}-\sigma\\ \cdot & \cdots & \cdot & \cdot\\ a_{n1} & \cdots & a_{n,n-1} & a_{nn}-\sigma\\ a_{n1} & \cdots & a_{n,n-1} & a_{nn}-\sigma\end{pmatrix}.$$

THEOREM 1. *The set of all row-semimagic squares of order n forms a non-semisimple algebra and has the radical $\mathfrak{N}$ such that $\mathfrak{N}^2=0$. The radical $\mathfrak{N}$ consists of all matrices of the type* (4), *and $\mathfrak{A}-\mathfrak{N}$ is the direct sum of a one-dimensional subalgebra and a simple subalgebra isomorphic to the total matric algebra of degree $n-1$. We can take a subalgebra $\mathfrak{B}$ such that a supplementary decomposition, $\mathfrak{A}=\mathfrak{B}+\mathfrak{N}$, holds* [3].

Obviously, we may form another decomposition like the above, so that such an algebra $\mathfrak{B}$ is not uniquely determined.

3. We now consider the semimagic squares defined in Section 1. In this case, the transformed matrix A' of (2) must satisfy

$$\sum_{i=1}^{n-1} a'_{ij} = \sigma(A) - na_{nj} \qquad (j = 1, \cdots, n-1). \tag{5}$$

In this section we first consider the case in which the characteristic p of the field Ω of elements of matrices does not divide the order n. Then, from (5), we have

$$a_{nj} = \sigma(A)/n - \sum_{i=1}^{n-1} a'_{ij}/n \qquad (j = 1, \cdots, n-1),$$

so that the matrix A' is completely determined by its elements a'_{ij} for $i, j = 1, \cdots, n-1$ and $a'_{nn}=\sigma(A)$. Therefore, the dimension of $\mathfrak{A}$ is $(n-1)^2+1$.

Now consider the matrix $Q=(q_{ij})$, $q_{ii}=1$ $(i=1, \cdots, n)$, $q_{nj}=-1/n$ $(j=1, \cdots, n-1)$, and $q_{ij}=0$ otherwise. When we transform A' by Q, then $A''=Q^{-1}A'Q$ is given by

$$A''=\begin{pmatrix}0 & \cdots & 0 & 0\\ \cdot & \cdots & \cdot & \cdot\\ 0 & \cdots & 0 & 0\\ 0 & \cdots & 0 & \sigma\end{pmatrix}+\begin{pmatrix}a'_{11} & \cdots & a'_{1,n-1} & 0\\ \cdot & \cdots & \cdot & \cdot\\ a'_{n-1,1} & \cdots & a'_{n-1,n-1} & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}.$$

Denote the set of all matrices of the first and second types by $\mathfrak{L}''$ and $\mathfrak{M}''$, respectively. Then, $\mathfrak{L}''$ and $\mathfrak{M}''$ are both two-sided ideals of $\mathfrak{A}''$, and $\mathfrak{A}''$ is the direct sum of $\mathfrak{L}''$ and $\mathfrak{M}''$. With reference to the algebra $\mathfrak{A}$, the ideal $\mathfrak{L}$ consists of all matrices for which $a_{ij}=a_{nj}=\sigma(A)/n$, namely, all elements are equal. The ideal $\mathfrak{M}$ consists of all matrices whose $\sigma(A)$ is 0. Thus we obtain the decomposition (1).

THEOREM 2. *In case $p \nmid n$, the algebra $\mathfrak{A}$ of all semimagic squares of order n is semisimple, and $\mathfrak{A}$ may be split into the direct sum of two ideals $\mathfrak{L}$ and $\mathfrak{M}$, where $\mathfrak{L}$ is a one-dimensional subalgebra and $\mathfrak{M}$ is a subalgebra isomorphic to the total matric algebra of degree $n-1$. This decomposition is that of* (1).

4. In the case when n is divisible by p, condition (5) becomes

$$\sum_{i=1}^{n-1} a'_{ij} = \sigma(A).$$

Therefore, in this case the algebra $\mathfrak{A}$ of semimagic squares of order n is isomorphic to the algebra of all matrices of the following type:

$$A' = \begin{pmatrix} a'_{11} & \cdots & a'_{1,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & \cdots & a'_{n-1,n-1} & 0 \\ a_{n1} & \cdots & a_{n,n-1} & \sigma \end{pmatrix},$$

where, after removing the last row and column, A' becomes a column-semimagic square of order $n-1$.

Now consider the matrix $R=(r_{ij})$, where $r_{ii}=1$ $(i=1, \cdots, n)$, $r_{1j}=-1$ $(j=2, \cdots, n-1)$, and $r_{ij}=0$ otherwise. Then we have

$$A'' = R^{-1}A'R = \begin{pmatrix} \sigma & 0 & \cdots & 0 & 0 \\ a'_{21} & a'_{22} - a'_{21} & \cdots & a'_{2,n-1} - a'_{21} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & a'_{n-1,2} - a'_{n-1,1} & \cdots & a'_{n-1,n-1} - a'_{n-1,1} & 0 \\ a_{n1} & a_{n2} - a_{n1} & \cdots & a_{n,n-1} - a_{n1} & \sigma \end{pmatrix}.$$

The set $\mathfrak{A}''$ of all these matrices $A''=(a''_{ij})$ forms an algebra isomorphic to $\mathfrak{A}$, and $\mathfrak{A}''$ consists of all matrices whose elements in the first row and in the last column are all 0 except for a''_{11} and a''_{nn}. Now, the set of all matrices of the first type in (6) is a two-sided ideal $\mathfrak{N}''$ of $\mathfrak{A}''$, and is nilpotent with $\mathfrak{N}''^3=0$, as may easily be verified.

$$(6) \qquad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a''_{21} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a''_{n-1,1} & 0 & \cdots & 0 & 0 \\ a''_{n1} & a''_{n2} & \cdots & a''_{n,n-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 & \cdots & 0 & 0 \\ 0 & a''_{22} & \cdots & a''_{2,n-1} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & a''_{n-1,2} & \cdots & a''_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & \sigma \end{pmatrix}.$$

Also, $\mathfrak{A}''-\mathfrak{N}''$ is isomorphic to the algebra consisting of all matrices of the second

type in (6). Thus $\mathfrak{A}''-\mathfrak{N}''$ is semisimple and $\mathfrak{N}''$ is the radical of $\mathfrak{A}''$. The set of all matrices of the second type in (6) forms a subalgebra $\mathfrak{B}''$ of $\mathfrak{A}''$, although not an ideal, and $\mathfrak{A}''=\mathfrak{B}''+\mathfrak{N}''$ is the supplementary sum of $\mathfrak{B}''$ and $\mathfrak{N}''$.

If we return to the algebra $\mathfrak{A}$, the matrices of $\mathfrak{N}$ satisfy the conditions:

$$\sigma(A) = 0 \text{ and } a'_{ij} = a_{ij} - a_{nj} = \delta_i \quad (i = 2, \cdots, n-1, j = 1, \cdots, n-1).$$

This second relation holds also for $i=1$ and for $j=n$, since, considering that $p \mid n$, first for $j=n$,

$$a_{in} = -\sum_{j=1}^{n-1} a_{ij} = -\sum_{j=1}^{n-1} a_{nj} - (n-1)\delta_i = a_{nn} + \delta_i,$$

and next for $i=1$,

$$a_{1j} = -\sum_{i=2}^{n} a_{ij} = -(n-1)a_{nj} - \sum_{i=2}^{n-1} \delta_i = a_{nj} - \sum_{i=2}^{n-1} \delta_i = a_{nj} + \delta_1,$$

where $\delta_1 = -\sum_{i=2}^{n-1} \delta_i$. Thus the ith row differs from the nth row by δ_i, for every $i=1, \cdots, n-1$, and $\sum_{i=1}^{n-1} \delta_i = 0$. If we put $a_{nj}=c_j+d_n$ for $j=1, \cdots, n$ and $d_i=\delta_i+d_n$ for $i=1, \cdots, n-1$, the matrix A may be written in the form

$$A = C + D = (c_{ij}) + (d_{ij}) = (c_{ij} + d_{ij}),$$

where

$$(7) \qquad C = (c_{ij}), \quad c_{ij} = c_j \ (i, j = 1, \cdots, n), \quad \sum_{j=1}^{n} c_j = 0,$$

$$(8) \qquad D = (d_{ij}), \quad d_{ij} = d_i \ (i, j = 1, \cdots, n), \quad \sum_{i=1}^{n} d_i = 0.$$

The sum of the set $\mathfrak{C}$ of all matrices C and the set $\mathfrak{D}$ of all matrices D proves to be the radical $\mathfrak{N}$ of $\mathfrak{A}$.

THEOREM 3. *In the case $p \mid n$, the algebra $\mathfrak{A}$ of semimagic squares of order n is non-semisimple and has the radical $\mathfrak{N}$ such that $\mathfrak{N}^3=0$. The radical $\mathfrak{N}$ consists of all linear combinations of all matrices* (7) *and* (8), *and $\mathfrak{A}-\mathfrak{N}$ may be split into the direct sum of a one-dimensional subalgebra and a simple subalgebra isomorphic to the total matric algebra of degree $n-2$.*

5. We see that these algebras give us suitable illustrations of the theory of non-semisimple algebras [2]. For example, consider a decomposition into the supplementary sum of indecomposable left ideals about the algebra $\mathfrak{A}$ of row-semimagic squares (Section 2).

Decompose the matrix A of $\mathfrak{A}$ into the sum $A=A_1+\cdots+A_{n-1}+A_n$, where, for $j=1, \cdots, n-1$, the matrix A_j has $(a_{1j} \cdots a_{nj})'$ as its jth column, $(-a_{1j} \cdots -a_{nj})'$ as its nth column, and 0 elsewhere. The matrix A_n has all its

elements 0 except for the elements in the nth column, which are equal to σ. Then, this leads to a decomposition of $\mathfrak{A}$ into the supplementary sum of left ideals $\mathfrak{A}=\mathfrak{l}_1+\cdots+\mathfrak{l}_{n-1}+\mathfrak{l}_n$.

For $j=1, \cdots, n-1$, let E_j denote the matrix which has 1 in its jth row and jth column, -1 in its jth row and nth column, and 0 elsewhere. Then, E_j is an idempotent of $\mathfrak{A}$ and the left ideal it generates is $\mathfrak{l}_j$. The idempotent E_n, with every element of the nth column equal to 1 and every other element equal to 0, generates the ideal $\mathfrak{l}_n$. Thus we have $\mathfrak{l}_j=\mathfrak{A}E_j$ for every j.

The algebra $\mathfrak{A}$ contains the identity matrix as the unit element, which we denote by E, and we have $E=E_1+\cdots+E_{n-1}+E_n$ and $E_iE_j=0$ $(i\neq j)$.

First we consider the left ideals $\mathfrak{l}_1, \cdots, \mathfrak{l}_{n-1}$. If A_j temporarily denotes the matrix with $(a_1 \cdots a_n)^t$ as its jth column, $(-a_1 \cdots -a_n)^t$ as its nth column, and 0 elsewhere, the correspondence $A_j \leftrightarrow A_i$ shows that $\mathfrak{l}_j$ and $\mathfrak{l}_i$ are isomorphic to each other. For each j, $\mathfrak{l}_j$ contains the left ideal $\mathfrak{N}E_j$. Since $\mathfrak{N}E_j$ consists of all matrices with $(c \cdots c)^t$ as the jth column, $(-c \cdots -c)^t$ as the nth column, and 0 elsewhere, it is a one-dimensional vector space over the field Ω, while $\mathfrak{l}_j$ is an n-dimensional vector space.

As for the last ideal $\mathfrak{l}_n$, we have $AE_n=\sigma(A)E_n$, so that $\mathfrak{l}_n$ is also a one-dimensional vector space.

Now, $E_jAE_j=(a_{jj}-a_{nj})E_j$ for $j=1, \cdots, n-1$. Therefore, the ring $E_j\mathfrak{A}E_j$ is isomorphic to the field Ω of elements of matrices. Then, of course, it is completely primary. Hence it follows that the non-nilpotent left ideal $\mathfrak{l}_j=\mathfrak{A}E_j$ is indecomposable [2].

Next, considering the residue-class ring $\overline{\mathfrak{A}}=\mathfrak{A}-\mathfrak{N}$, we have $\overline{\mathfrak{A}}=\bar{\mathfrak{l}}_1+\cdots+\bar{\mathfrak{l}}_{n-1}+\bar{\mathfrak{l}}_n$, where, as an $\mathfrak{A}$-left space, $\bar{\mathfrak{l}}_j$ is isomorphic to $\mathfrak{A}E_j/\mathfrak{N}E_j$ for $j=1, \cdots, n-1$, and $\bar{\mathfrak{l}}_n$ to $\mathfrak{l}_n$. Accordingly, all ideals $\bar{\mathfrak{l}}_j$ are irreducible, $\bar{\mathfrak{l}}_1, \cdots, \bar{\mathfrak{l}}_{n-1}$ are mutually isomorphic, and $\bar{\mathfrak{l}}_1+\cdots+\bar{\mathfrak{l}}_{n-1}$ forms a simple subring of $\overline{\mathfrak{A}}$. If, for $j=1, \cdots, n-1$, we take $\bar{\mathfrak{l}}_j$ as an $\mathfrak{A}$-left space, we obtain the representation of $\mathfrak{A}$ as the total matric algebra of degree $n-1$. Also, if we take $\mathfrak{l}_n$ as an $\mathfrak{A}$-left space, we obtain the representation $A\rightarrow\sigma(A)$ of degree 1.

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THE SOLUTION OF LINEAR VECTOR DIFFERENTIAL EQUATIONS CONTAINING GYROSCOPIC TERMS

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1. Introduction. The purpose of this paper is to show how the solution of vector differential equations of the type governing the motion of the bob of a Foucault's pendulum and the motion of an electron in an electrostatic and a uniform magnetic field can be obtained deductively in vector form.

In his *Vectorial Mechanics*, Milne [2] determined the motion of the bob by a process of trial, seeking solutions in terms of rotating vectors. In a recent note to the *Mathematical Gazette*, Easthope [1] showed how the equation could be reduced to the standard simple-harmonic form by referring the motion to suitable rotating axes. The hint for this procedure was given by Milne (*loc. cit.*) and carried out in succeeding applications [3, 4].

Milne did not make the final step of integration, by which the position vector is given as a function of the time, but synthesized [5] it from the already known Cartesian components of rotating vectors. It is shown below that the final result can be obtained deductively by operator methods, after the choice of a suitable reference triad of unit vectors.

Essentially, the problem is a case of finding the solution of a special system of three ordinary linear differential equations in three dependent variables, the components of a vector function $\mathbf{r}$ of the scalar independent variable t; the special character is determined by the presence of a vector-product term, such as is frequently encountered in mechanics. These equations are written in vector-tensor form, in which the derivatives occur as scalar polynomial functions of the operator $D \equiv d/dt$. The differential equations then have the formal appearance of an algebraic system of tensor relations, which suggests an application of the well known theory of characteristic vectors. It turns out that there are three distinct eigenvalues, two of which are complex and conjugate. It is shown that the corresponding eigenvectors (one real and two conjugate complex) are mutually orthogonal. The tensor differential relations are thus reduced to three ordinary differential equations, separately involving the components of $\mathbf{r}$ in the directions of the eigenvectors. After these have been solved by standard operator methods, the solution can be reconstituted into a form independent of this complex reference triad of vectors. The paper concludes with an application of the method to the physical problem that stimulated the investigation.

2. The general inhomogeneous equation. The equation to be considered has the form

$$f(D)\mathbf{r} + 2\boldsymbol{\omega} \times g(D)\mathbf{r} = \mathbf{F}(t), \tag{1}$$

where f and g are polynomial functions having real coefficients, $D \equiv d/dt$, $\boldsymbol{\omega} = \omega\mathbf{k}$ is a constant vector having the direction of the unit vector $\mathbf{k}$, $\mathbf{F}$ a prescribed real vector function of t and $\mathbf{r}$ a function of t, to be determined.

2.1 The complementary function. The second term on the left side of (1), written as the vector product of an axial and a polar vector, may, by introducing the unit tensor $\mathbf{U}$, also be written as the inner product of a proper tensor and the polar vector. Then the homogeneous equation becomes

$$f(D)\mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{U}g(D)\cdot\mathbf{r} = 0, \tag{2}$$

whose solution may be facilitated by seeking directions in which the second term is parallel to the first, *i.e.* the eigenvectors $\mathbf{e}_r$, $r=1, 2, 3$, of the tensor operator $2\boldsymbol{\omega}\times\mathbf{U}g(D)$, corresponding to eigenvalues λ_r. The vector equation may then be resolved into three scalar equations which separately determine the components of $\mathbf{r}$ in the directions of the eigenvectors.

Since $\mathbf{r}=\mathbf{U}\cdot\mathbf{r}$, the latter are the solutions of the equation

$$\left| 2\boldsymbol{\omega} \times \mathbf{U}g - \mathbf{U}\lambda \right| = 0. \tag{3}$$

In terms of Cartesian components of $\boldsymbol{\omega}$ this becomes

$$\begin{vmatrix} -\lambda & -2\omega_3 g & 2\omega_2 g \\ 2\omega_3 g & -\lambda & -2\omega_1 g \\ -2\omega_2 g & 2\omega_1 g & -\lambda \end{vmatrix} = 0,$$

and the corresponding components of the eigenvectors are then proportional to the cofactors of, say, the first row of this determinant, *viz.*

$$(\lambda_r^2 + 4\omega_1^2 g^2,\ 4\omega_1\omega_2 g^2 + 2\lambda_r\omega_3 g,\ 4\omega_3\omega_1 g^2 - 2\lambda_r\omega_2 g).$$

The cubic in λ yields the eigenvalues $\lambda_r=0,\ \pm 2i\omega g$, and the corresponding eigenvectors are proportional to

$$(\omega_1,\ \omega_2,\ \omega_3),\ (\omega_1^2 - \omega^2,\ \omega_1\omega_2 \pm i\omega\omega_3,\ \omega_3\omega_1 \mp i\omega\omega_2).$$

The first vector is parallel to $\mathbf{k}$ and the other two are complex and conjugate; they are all mutually orthogonal, in the sense that the three unit vectors $\mathbf{e}_r$ parallel to these directions are such that

$$\mathbf{e}_r\cdot\mathbf{e}_s^* = \mathbf{e}_r^*\cdot\mathbf{e}_s = \delta_{rs}, \tag{4}$$

where an asterisk denotes a complex conjugate and δ_{rs} is the usual Kronecker symbol.

If we choose the Cartesian axes so that

$$\mathbf{k} = (0, 0, 1), \tag{5}$$

the eigenvectors have the simple form*

* The significance of their association with the Cartesian 1- and 2-axes will become apparent below.

$$\mathbf{e}_1 = (1, i, 0)/\sqrt{2}, \qquad \mathbf{e}_2 = (1, -i, 0)/\sqrt{2}, \tag{6}$$
$$\mathbf{e}_3 = (0, 0, 1) = \mathbf{k},$$

corresponding to the eigenvalues

$$\lambda_1 = -2i\omega g, \qquad \lambda_2 = 2i\omega g, \qquad \lambda_3 = 0. \tag{7}$$

Any vector $\mathbf{V}$ can be expressed in terms of components† $\overline{V}_r$ such that

$$\overline{V} = \mathbf{V}_r\mathbf{e}_r, \tag{8}$$

where summation is implied by the repeated subscript. Then, by virtue of (4),

$$\overline{V}_r = \mathbf{V}\cdot\mathbf{e}_r^*. \tag{9}$$

Using (8) and (9), scalar and vector products and the various derivatives of the vector calculus can be expressed in terms of these components. The two component vectors $\mathbf{V}_\parallel$ and $\mathbf{V}_\perp$, parallel and perpendicular to $\mathbf{k}$, are then

$$\mathbf{V}_\parallel = \overline{V}_3\mathbf{e}_3, \qquad \mathbf{V}_\perp = \overline{V}_1\mathbf{e}_1 + \overline{V}_2\mathbf{e}_2. \tag{10}$$

When the Cartesian axes are chosen as in (5), so that $\mathbf{V} = (V_1, V_2, V_3)$, the components $\overline{V}_r$ take the simple form

$$\overline{V}_1 = (V_1 - iV_2)/\sqrt{2}, \qquad \overline{V}_2 = (V_1 + iV_2)/\sqrt{2}, \qquad \overline{V}_3 = V_3. \tag{11}$$

If now the Cartesian 1- and 2-axes are regarded as real and imaginary axes on an Argand plane, $\sqrt{2}\overline{V}_2$ represents the vector $\mathbf{V}_\perp$ and $\sqrt{2}\overline{V}_1$ its reflection in the 1-axis. Such combinations have long been used in the contexts of dynamics and magneto-optics, and more recently [6], as explicit components in the magneto-ionic theory.

Since the tensor operator may now be written

$$2\boldsymbol{\omega} \times \mathbf{U}g = \sum_a \lambda_a \mathbf{e}_a\mathbf{e}_a^*, \tag{12}$$

the homogeneous equation (2) becomes, for $\mathbf{r} = \bar{x}_r\mathbf{e}_r$,

$$\{f(D) - 2i\omega g(D)\}\bar{x}_1\mathbf{e}_1 + \{f(D) + 2i\omega g(D)\}\bar{x}_2\mathbf{e}_2 + f(D)\bar{x}_3\mathbf{e}_3 = 0. \tag{13}$$

Moreover, the unit vectors $\mathbf{e}_r$ are independent, so that (13) gives three separate scalar differential equations which determine $\bar{x}_1$, $\bar{x}_2$, to as many arbitrary constants as the order of the higher of the polynomials $f(D)$, $g(D)$, and $\bar{x}_3$ to as many as the order of $f(D)$. If such solutions are denoted by $\bar{x}_r = u_r(t)$, then the complementary function of the equation (1) is

$$\mathbf{r} = \mathbf{u}(t) = u_r(t)\mathbf{e}_r. \tag{14}$$

† Since the reciprocal set of vectors $\mathbf{e}^1 = \mathbf{e}_2$, $\mathbf{e}^2 = \mathbf{e}_1$, $\mathbf{e}^3 = \mathbf{e}_3$, there is no point in distinguishing between covariant and contravariant components.

It is possible, however, to reconstitute the right side of (14) into a form that is independent of any reference triad of vectors. Any of the linearly independent particular solutions of the homogeneous equation in $\bar{x}_1$ can be written in the form $x_s - iy_s$ where $x_s(t)$, $y_s(t)$ are real. There will then be a corresponding particular solution of the equation in $\bar{x}_2$, *viz.* $x_s + iy_s$,* and thus

$$\begin{aligned} \mathbf{u}_\perp(t) &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \\ &= A_{1s}(x_s - iy_s)\mathbf{e}_1 + A_{2s}(x_s + iy_s)\mathbf{e}_2, \end{aligned} \tag{15}$$

where the A_{1s}, A_{2s} are the constants of integration, and s ranges from 1 to the order of the equation. Now, using (6), it is not difficult to show that

$$\mathbf{e}_3 \times \mathbf{e}_1 = -i\mathbf{e}_1, \qquad \mathbf{e}_3 \times \mathbf{e}_2 = i\mathbf{e}_2. \tag{16}$$

Hence (15) may be written as the sum

$$\mathbf{u}_\perp(t) = \mathbf{A}_s x_s + \mathbf{k} \times \mathbf{A}_s y_s, \tag{17}$$

where

$$\mathbf{A}_s = A_{1s}\mathbf{e}_1 + A_{2s}\mathbf{e}_2$$

are constant vectors in the plane perpendicular to $\mathbf{k}$, determined by the initial conditions. Thus the real and imaginary parts of u_1 or u_2 provide Cartesian components of $\mathbf{u}$ in the plane perpendicular to $\mathbf{k}$. It will be seen in Section 3 that in the problem of Foucault's pendulum these solutions take the form of uniformly rotating vectors of constant amplitude, as given by Milne's trial solutions [2].

The third Cartesian component may be written as

$$\mathbf{u}_\parallel(t) = u_3\mathbf{e}_3 = C_s z_s \mathbf{k}, \tag{18}$$

where the z_s are linearly independent solutions of the homogeneous equation in $\bar{x}_3$ and the C_s are constants of integration. Thus the complementary function of the equation (1) becomes

$$\mathbf{u} = \mathbf{A}_s x_s + \mathbf{k} \times \mathbf{A}_s y_s + \mathbf{k} C_s z_s, \tag{19}$$

a sum containing the requisite number of scalar constants of integration.

2.2 The particular integral. By resolving $\mathbf{F}(t)$ into components, as in (8) and (9), the inhomogeneous equation (1) yields the three scalar equations

$$\begin{aligned} \{f(D) - 2i\omega g(D)\}\bar{x}_1 &= \overline{F}_1, \\ \{f(D) + 2i\omega g(D)\}\bar{x}_2 &= \overline{F}_2, \\ f(D)\bar{x}_3 &= \overline{F}_3. \end{aligned} \tag{20}$$

If particular integrals of these are $\bar{x}_r = \phi_r(t)$,

$$\mathbf{r} = \boldsymbol{\phi}(t) = \phi_r(t)\mathbf{e}_r \tag{21}$$

is a particular integral of (1), whose general solution can be written

* This follows because the operations $f(D)$, $g(D)$ on real functions result in real functions.

$$\mathbf{r} = \mathbf{u}(t) + \boldsymbol{\phi}(t). \tag{22}$$

Again, the right side of (21) can be reconstituted independently of the triad $\mathbf{e}_r$. Since $\overline{F}_1$, $\overline{F}_2$ are complex conjugates, it follows that if $\sqrt{2}\phi_1 = \xi - i\eta$ then $\sqrt{2}\phi_2 = \xi + i\eta$, so that, by (16)

$$\begin{aligned} \boldsymbol{\phi}_\perp &= \phi_1\mathbf{e}_1 + \phi_2\mathbf{e}_2 \\ &= \xi\mathbf{i} + \eta\mathbf{k} \times \mathbf{i}, \end{aligned} \tag{23}$$

where

$$\mathbf{i} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2},$$

is an arbitrary unit vector perpendicular to $\mathbf{k}$; $\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}$ have respectively the directions of the real and imaginary axes of the Argand plane referred to above. Then, if $\zeta(t) \equiv \phi_3(t)$,

$$\boldsymbol{\phi}_\parallel = \zeta\mathbf{k} \tag{24}$$

and

$$\boldsymbol{\phi} = \xi\mathbf{i} + \eta\mathbf{k} \times \mathbf{i} + \zeta\mathbf{k}, \tag{25}$$

is a particular integral of (1). The general solution is then given by (22) with (19) and (25).

3. Example. The undamped motion of a Foucault's pendulum bob, and that of an elastically bound electron in a uniform magnetic field, are covered by taking

$$f(D) \equiv D^2 + n^2, \qquad g(D) \equiv D. \tag{26}$$

Then it can be seen from (13) that the function $u_1(t)$ is the general solution of the equation

$$\{D - i\omega + i\sqrt{(n^2 + \omega^2)}\}\{D - i\omega - i\sqrt{(n^2 + \omega^2)}\}\bar{x}_1 = 0, \tag{27}$$

viz.

$$u_1 = e^{i\omega t}(A_1 e^{-i\sqrt{(n^2+\omega^2)}\,t} + B_1 e^{i\sqrt{(n^2+\omega^2)}\,t}), \tag{28}$$

where A_1 and B_1 are arbitrary constants.* It can be seen also from the above interpretation of the vector components that the two terms in (28) represent vectors of constant amplitude rotating about the direction of $\mathbf{k}$ with constant angular velocities in a right-handed and a left-handed sense, respectively. The locus of the resultant is a Lissajous figure traced out on a plane perpendicular to $\mathbf{k}$. Referred to a frame of reference which itself rotates with angular velocity $-\omega$, the angular velocities of the two vectors are $\pm\sqrt{(n^2+\omega^2)}$ and the locus is an ellipse.

* In the problem of Foucault's pendulum it is proper to neglect ω^2 as compared to n^2 in (28) and in the succeeding formulas.

Similarly,

$$u_2 = e^{-i\omega t}(A_2 e^{i\sqrt{(n^2+\omega^2)}\,t} + B_2 e^{-i\sqrt{(n^2+\omega^2)}\,t}), \tag{29}$$

represents two vectors of constant amplitude rotating respectively with the same angular velocities as in (28), giving again an ellipse when referred to a frame rotating with angular velocity $-\boldsymbol{\omega}$. Finally,

$$u_3 = C\cos nt + D\sin nt \tag{30}$$

represents a linear oscillation of period $2\pi/n$ along the direction of $\mathbf{k}$.

Now the separate terms of (28) or (29) give

$$\begin{aligned} x_1 &= \cos\{\sqrt{(n^2+\omega^2)} - \omega\}t,\ y_1 = \sin\{\sqrt{(n^2+\omega^2)} - \omega\}t, \\ x_2 &= \cos\{\sqrt{(n^2+\omega^2)} + \omega\}t,\ y_2 = -\sin\{\sqrt{(n^2+\omega^2)} + \omega\}t. \end{aligned} \tag{31}$$

Hence, by (18), (19), and (30) the complementary function may be written

$$\begin{aligned} \mathbf{u}(t) = {} & \mathbf{A}\cos\{\sqrt{(n^2+\omega^2)} - \omega\}t + \mathbf{k}\times\mathbf{A}\sin\{\sqrt{(n^2+\omega^2)} - \omega\}t \\ & + \mathbf{B}\cos\{\sqrt{(n^2+\omega^2)} + \omega\}t - \mathbf{k}\times\mathbf{B}\sin\{\sqrt{(n^2+\omega^2)} + \omega\}t \\ & + \mathbf{k}(C\cos nt + D\sin nt), \end{aligned} \tag{32}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary constant vectors perpendicular to $\mathbf{k}$, and C and D are arbitrary scalar constants. The first two pairs of terms each represent vectors of constant amplitudes A, B, again rotating about $\mathbf{k}$ with angular velocities $\pm\sqrt{(n^2+\omega^2)} - \omega$ respectively.

In the present case, the component $\phi_1(t)$ of the particular integral is, as usual, given by

$$2i\sqrt{(n^2+\omega^2)}\,\phi_1 = \left[\frac{1}{D - i\omega - i\sqrt{(n^2+\omega^2)}} - \frac{1}{D - i\omega + i\sqrt{(n^2+\omega^2)}}\right]\overline{F}_1,$$

whence

$$\phi_1 = \frac{1}{\sqrt{(n^2+\omega^2)}}\int^t \overline{F}_1(\tau)e^{i\omega(t-\tau)}\sin\{\sqrt{(n^2+\omega^2)}(t-\tau)\}\,d\tau. \tag{33}$$

Similarly,

$$\phi_2 = \frac{1}{\sqrt{(n^2+\omega^2)}}\int^t \overline{F}_2(\tau)e^{-i\omega(t-\tau)}\sin\{\sqrt{(n^2+\omega^2)}(t-\tau)\}\,d\tau, \tag{34}$$

and

$$\phi_3 = \frac{1}{n}\int^t \overline{F}_3(\tau)\sin n(t-\tau)\,d\tau. \tag{35}$$

The required particular integral is now obtained by substitution from these

results into (21). This can be expressed in a form independent of any coordinate system, directly as in (15) and (17), or from (25). The result is

$$\phi(t) = \frac{1}{\sqrt{(n^2+\omega^2)}} \int^t \{\mathbf{F}_\perp(\tau)\cos\omega(t-\tau) - \mathbf{k}\times\mathbf{F}_\perp(\tau)\sin\omega(t-\tau)\} \times \sin\{\sqrt{(n^2+\omega^2)}(t-\tau)\}d\tau + \frac{1}{n}\int^t F_\parallel(\tau)\sin n(t-\tau)d\tau. \tag{36}$$

Finally, the constants $\mathbf{A}$, $\mathbf{B}$, C, D can be evaluated for the most general initial conditions,

$$\mathbf{r} = a\mathbf{i} + b\mathbf{k}, \qquad D\mathbf{r} = u\mathbf{j} + v\mathbf{k}, \tag{37}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors perpendicular to $\mathbf{k}$. The result is

$$\mathbf{A} = \frac{a\{\omega + \sqrt{(n^2+\omega^2)}\}\mathbf{i} + u\mathbf{j}\times\mathbf{k}}{2\sqrt{(n^2+\omega^2)}},$$

$$\mathbf{B} = -\frac{a\{\omega - \sqrt{(n^2+\omega^2)}\}\mathbf{i} + u\mathbf{j}\times\mathbf{k}}{2\sqrt{(n^2+\omega^2)}}, \tag{38}$$

$$C = b, \qquad D = v/n,$$

with the lower limit 0 in the integrals on the right side of (36). The general solution is then given by (22) with (32) and (36).

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CORRECTION

W. E. Bleick, *Fourier analysis of engine unbalance by contour integration*, this MONTHLY, vol. 63, 1956, pp. 466–472. Equation (17), p. 469 should read "$b_1 = 4[E(\alpha^2) - (1-\alpha^2)K(\alpha^2)]/\pi\alpha$."

MATHEMATICAL NOTES

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Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

A GENERALIZATION OF THE SINE FUNCTION

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One generalization of the elementary circular and hyperbolic functions consists in finding fundamental sets of solutions of the equations $d^ny/dx^n \pm y = 0$, having properties analogous to those of the elementary functions. These functions are variously referred to as higher-order sine functions, Olivier functions, or Villarceau functions. Their history begins with the work of V. Riccati [1] and Wronski [2]. Günther [3] gives an excellent survey of the relevant literature to 1880, and Poli [4] provides a more recent summary. A thorough bibliography will appear in a forthcoming note [5]. For accessible accounts of these functions the reader is referred to papers by Humbert [6], Ward [7], Dramba [8], Oniga [9], and Silberstein [10].

Brodetsky [11] has made a generalization in a different direction by the following considerations relative to the second order equation $d^2y/dx^2 = I(x)y$. Let functions $C(x, \xi)$ and $S(x, \xi)$ be defined such that if $y(x)$ is a solution of the equation then $y(x+\xi) = C(x, \xi)y(x) + S(x, \xi)y'(x)$. The C and S functions have certain properties analogous to those of the elementary circular and hyperbolic functions. In the present note Brodetsky's analysis is extended to the nth order equation

$$d^ny/dx^n = I(x)y, \tag{1}$$

where $I(x)$ is a given continuous function defined on an interval (a, b). The new functions introduced in this way include as special cases the above two generalizations.

Define functions $C_j \equiv C_j(x, \xi)$, $(j = 1, \cdots, n)$, such that if $y(x)$ is a solution of (1) then $y(x+\xi) = \sum_{j=1}^{n} C_j y^{(j-1)}(x)$, where $y^{(j)}(x) \equiv d^jy/dx^j$, $y^{(0)}(x) \equiv y(x)$. Since

$$\frac{\partial y(x+\xi)}{\partial \xi} = \sum_{j=1}^{n} \frac{\partial C_j}{\partial \xi}\, y^{(j-1)}(x),$$

$$\frac{\partial y(x+\xi)}{\partial x} = \sum_{j=1}^{n} \frac{\partial C_j}{\partial x}\, y^{(j-1)}(x) + \sum_{j=1}^{n} C_j y^{(j)}(x),$$

and $\partial y(x+\xi)/\partial x = \partial y(x+\xi)/\partial \xi$ it follows that

$$\left(\frac{\partial C_1}{\partial \xi} - \frac{\partial C_1}{\partial x} - I(x)C_n\right)y(x) + \sum_{j=2}^{n}\left(\frac{\partial C_j}{\partial \xi} - \frac{\partial C_j}{\partial x} - C_{j-1}\right)y^{(j-1)}(x) = 0.$$

This relation must hold for every linearly independent solution of (1), whence

$$\frac{\partial C_1}{\partial \xi} - \frac{\partial C_1}{\partial x} = I(x)\ C_n, \qquad \frac{\partial C_j}{\partial \xi} - \frac{\partial C_j}{\partial x} = C_{j-1} \quad (j = 2, \cdots, n). \tag{2}$$

The relation of the functions $C_j(x, \xi)$ to the differential equation (1) is obtained by differentiating $y(x+\xi)$ n times with respect to ξ. Thus

$$\frac{\partial^n(y+\xi)}{\partial \xi^n} = \sum_{j=1}^{n}\frac{\partial^n C_j}{\partial \xi^n}\,y^{(j-1)}(x)$$

$$= I(x+\xi)y(x+\xi) = I(x+\xi)\sum_{j=1}^{n} C_j y^{(j-1)}(x),$$

whence

$$\partial^n C_j/\partial \xi^n = I(x+\xi)C_j \qquad (j = 1, \cdots, n). \tag{3}$$

By substituting $\xi = 0$ in $y(x+\xi)$ and its derivatives with respect to ξ, we obtain

$$\frac{\partial^i C_j(x, 0)}{\partial \xi^i} = \begin{cases} 1, & i = j-1 \\ 0, & i \neq j-1 \end{cases} \qquad (i = 0, \cdots, n-1; j = 1, \cdots, n), \tag{4}$$

which are the analogues of $\cos 0 = 1$, $\sin 0 = 0$, *etc.* Since $\partial^n C_i/\partial \xi^n = I(x+\xi)C_i$, $\partial^n C_j/\partial \xi^n = I(x+\xi)C_j$, it follows that

$$\left(\frac{\partial^n C_i}{\partial \xi^n}\right)C_j = \left(\frac{\partial^n C_j}{\partial \xi^n}\right)C_i \qquad (i, j = 1, \cdots, n;\ i \neq j).$$

In Brodetsky's terminology, the functions C_j are the principal integrals of equation (1) in the variable ξ with shifted origin. Hence the solution of (1) with argument $x_0+\xi$ is obtained from the principal integrals C_j of (1) in ξ with origin at $x = x_0$ by multiplying them respectively by the values of $y, y', \cdots, y^{(n-1)}$ at $x = x_0$, and forming the sum $\sum_{j=1}^{n} C_j(x_0, \xi)y^{(j-1)}(x_0)$.

An addition formula is obtained by expanding $y(x+\xi+\eta)$ in two ways. Thus

$$y(x+\xi+\eta) = \sum_{j=1}^{n} C_j(x, \xi+\eta)y^{(j-1)}(x),$$

$$y(x+\xi+\eta) = \sum_{j=1}^{n} C_j(x+\xi, \eta)y^{(j-1)}(x+\xi)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} C_j(x+\xi, \eta)\frac{\partial^{j-1}C_k(x, \xi)}{\partial \xi^{j-1}}\,y^{(k-1)}(x),$$

whence

$$C_j(x, \xi + \eta) = \sum_{i=1}^{n} \left(\frac{\partial^{i-1} C_j(x, \xi)}{\partial \xi^{i-1}} \right) C_i(x + \xi, \eta) \quad (j = 1, \cdots, n). \tag{5}$$

If $I(x) =$ constant (in particular, $I(x) = \pm 1$), the functions C_j do not depend on x, and (5) reduces to the addition formula for the Olivier functions.

If a Taylor series expansion is assumed for $I(x)$, the following series representations of $C_j(x, \xi)$ are obtained:

$$C_j(x, \xi) = \sum_{k=0}^{\infty} A_{kj}(x) \frac{\xi^k}{k!} \qquad (j = 1, \cdots, n), \tag{6}$$

where

$$A_{kj}(x) = \begin{cases} 0, & k \neq j - 1 \\ 1, & k = j - 1 \end{cases} \qquad (k = 0, \cdots, n - 1; j = 1, \cdots, n).$$

The recursion relations for A_{kj}, $(k \geqq n)$, are given by

$$A_{n+s,j} = I^{(s)} A_{0j} + sI^{(s-1)} A_{1j} + [s(s-1)/2!] I^{(s-2)} A_{2j} + \cdots + IA_{sj}$$
$$(j = 1, \cdots, n; s = 0, 1, \cdots),$$

where $A_{kj} \equiv A_{kj}(x)$, $I^{(r)} \equiv d^r I / dx^r$. A detailed analysis based on the Taylor expansion for the case $n = 2$ will be found in Brodetsky's paper.

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A NOTE ON NEIGHBORING JORDAN CURVES

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One of the problems in conformal mapping is to find conditions on two Jordan curves C and C' so that the difference between the functions which map the unit circle conformally onto the interiors of the two curves will be small in some sense. One of the conditions sometimes assumed is that each curve lies within a certain neighborhood of the other. Under certain additional conditions this assumption is redundant, because if C' lies within an ϵ neighborhood of C, it can be proved that C lies within an ϵ_1 neighborhood of C', where $\epsilon_1 = O(\epsilon)$ as $\epsilon \to 0$. We give an example of a theorem of this type.

Let C and C' be Jordan curves in the complex z plane satisfying the conditions:

(H_1) The interior of C' contains the origin $z=0$.

(H_2) The interior of C contains the circle $|z| = b > 0$.

(H_3) If z_1 and z_2 are on C, then one of the arcs of C connecting z_1 and z_2 lies within a circle of diameter $d = c|z_1 - z_2|$, where c is a fixed constant (necessarily >1).

We now show that if C' lies within an ϵ neighborhood of C, where $\epsilon < b/2c$, then C lies within an ϵ_1 neighborhood of C' if $\epsilon_1 > (2c+1)\epsilon$.

An arc $\overline{z_1 z_2}$ of C which has the property assumed in (H_3) will be called a minimal arc joining z_1 and z_2. If D is the diameter of C and $|z_1 - z_2| < D/2c$, then only one of the two arcs joining z_1 and z_2 can be minimal.

Let $z_1', z_2', \cdots, z_n' = z_1'$ be points of C', ordered according to the orientation of C' and chosen so that each arc $\overline{z_k' z_{k+1}'}$ ($k = 1, 2, \cdots, n-1$) is contained in a circle of diameter $\delta < b/c - 2\epsilon$. For each point z_k' let z_k be a corresponding point of C such that $|z_k' - z_k| < \epsilon$ and $z_n = z_1$. Let γ_k be the minimal arc of C from z_k to z_{k+1} and let Γ be the curve formed by joining the arcs $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ end to end. Considered as point sets, Γ is a subset of C. The variation in arg z over Γ, denoted by $\Delta_\Gamma \arg z$, is a multiple of 2π since the initial and terminal points of Γ coincide.

Let $z = z(t)$ and $z = z'(t)$, $0 \leqq t \leqq 1$, be continuous parametric representations of Γ and C' respectively, with $z_k = z(t_k)$, $z_k' = z'(t_k')$, $0 = t_1 \leqq t_2 \leqq \cdots \leqq t_n = 1$, $0 = t_1' \leqq t_2' \leqq \cdots \leqq t_n' = 1$. Let $a(t)$ and $a'(t)$ be values of arg $z(t)$ and arg $z'(t)$ respectively, each continuous for $0 \leqq t \leqq 1$ and chosen so that $|a(0) - a'(0)| < \pi$. This can be done since $|z_1| > b$ and $|z_1 - z_1'| < \epsilon < b/2$ so that the line segment from z_1 to z_1' cannot contain the origin.

Let us assume that $|a(t_k) - a'(t_k')| < \pi$ for some k ($0 \leqq k < n$). The arcs $\overline{z_k z_{k+1}}$ and $\overline{z_k' z_{k+1}'}$ are both contained in a circle S of radius $c(2\epsilon + \delta)$ about z_k as center, and S does not contain the origin since $c(2\epsilon + \delta) < b$. Let $A(z)$ be a value of arg z for $z \in S$ defined so as to be continuous in S and so that $A(z_k) = a(t_k)$. Since $|A(z_k) - A(z_k')| < \pi$, $|a(t_k) - a'(t_k')| < \pi$, and $A(z_k') - a'(t_k')$ is a multiple of 2π,

$A(z_k')=a'(t_k')$, and from continuity $A(z_{k+1})=a(t_{k+1})$, $A(z_{k+1}')=a'(t_{k+1}')$. Hence, $|a(t_{k+1})-a'(t_{k+1}')|<\pi$.

By induction $|a(t_k)-a'(t_k')|<\pi$ for $1\leqq k\leqq n$. Hence $|\Delta_\Gamma \arg z-\Delta_{\sigma'} \arg z|$ is less than 2π and therefore zero, so $\Delta_\Gamma \arg z=\Delta_{\sigma'} \arg z=2\pi$ and Γ cannot be a proper subset of C. Any point of C is also on Γ and lies within a distance $c(2\epsilon+\delta)$ of one of the points z_k, hence within a distance $c(2\epsilon+\delta)+\epsilon$ of a point of C'. By choosing δ sufficiently small this latter distance can be made less than ϵ_1 if $\epsilon_1>(2c+1)\epsilon$.

EXTENSIONS OF THE LAW OF THE MEAN

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An extension of the law of the mean was published by Karamata [2] and the following year Vučković [4] gave an extension of Karamata's result. Utz [3] pointed out that both authors stated a stronger conclusion than can be secured. The purpose of this note is to show that Vučković's result may be obtained directly from a theorem of Kametani [1]. Karamata's result follows as a special case.

Theorem (*Vučković*). *If the functions f and g are continuous for $a\leqq x\leqq b$ and if the right and left hand derivatives f'_+, f'_-, g'_+, g'_- exist on $a<x<b$, then there exist numbers x_0, p, q with $a<x_0<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$ for which* $[pf'_+(x_0)+qf'_-(x_0)]\cdot[g(b)-g(a)]=[pg'_+(x_0)+qg'_-(x_0)][f(b)-f(a)]$.

Proof. Let $a\leqq x\leqq x'\leqq b$, let I be the closed interval $[x, x']$, let $\phi(I)=f(x')-f(x)$ and let $\sigma(I)=g(x')-g(x)$. Then ϕ and σ are additive continuous functions of an interval on the closed interval $I_0=[a, b]$. Hence [1, p. 2] there exist a point x_0 and a descending sequence of intervals $\{I_n\}=\{[a_n, b_n]\}$ with the following properties:

(1) $I_0\supset I_1\supset I_2\cdots\ni x_0$,

(2) $\lim_{n\to\infty}(b_n-a_n)=0$,

(3) x_0 is an interior point of each I_n,

(4) $[f(b_n)-f(a_n)][g(b)-g(a)]=[g(b_n)-g(a_n)][f(b)-f(a)]$ $(n=1, 2, \cdots)$.

Since $a_n<x_0<b_n$ $(n=1, 2, \cdots)$, $b_n-a_n\neq 0$ and we can divide each side of (4) by b_n-a_n which gives

$$\left[\frac{f(b_n)-f(a_n)}{b_n-a_n}\right][g(b)-g(a)]=\left[\frac{g(b_n)-g(a_n)}{b_n-a_n}\right][f(b)-f(a)] \qquad (n=1, 2, \cdots).$$

This may be written

$$\left[\frac{f(b_n)-f(x_0)}{b_n-x_0}\,\frac{b_n-x_0}{b_n-a_n}+\frac{f(x_0)-f(a_n)}{x_0-a_n}\,\frac{x_0-a_n}{b_n-a_n}\right][g(b)-g(a)]$$

$$=\left[\frac{g(b_n)-g(x_0)}{b_n-x_0}\,\frac{b_n-x_0}{b_n-a_n}+\frac{g(x_0)-g(a_n)}{x_0-a_n}\,\frac{x_0-a_n}{b_n-a_n}\right][f(b)-f(a)]. \tag{1}$$

Since $0<(b_n-x_0)/(b_n-a_n)<1$ $(n=1, 2, \cdots)$, there exist a number p, $0\leqq p\leqq 1$, and a subsequence $\{I_{n'}\}$ of the sequence $\{I_n\}$ such that $\lim_{n'\to\infty}(b_{n'}-x_0)/(b_{n'}-a_{n'})=p$. Also, $\lim_{n'\to\infty}(x_0-a_{n'})/(b_{n'}-a_{n'})=q$ and $p+q=1$. Hence replacing n by n' in (1) and then taking the limit of both sides as $n'\to\infty$, we obtain $[pf'_+(x_0)+qf'_-(x_0)][g(b)-g(a)]=[pg'_+(x_0)+qg'_-(x_0)][f(b)-f(a)]$ with $p\geqq 0$, $q\geqq 0$, $p+q=1$, which completes the proof.

If we set $g(x)=x$ in the above theorem, we obtain $[pf'_+(x_0)+qf'_-(x_0)][b-a]=f(b)-f(a)$, which is Karamata's result.

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A DETERMINANT

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Put

$$D_n=\left|\binom{n+1}{2s-r}\right| \qquad (r, s=0, 1, \cdots, n), \tag{1}$$

a determinant of order $n+1$. The interesting result

$$D_n=2^{n(n+1)/2} \tag{2}$$

was stated without proof by Nesbitt [1, p. 316]. Two proofs of (2) have been given by Niblett [2].

It may be of interest to point out that this result is a special case of a more general result that can be proved rather easily. We observe first that

$$D_n=D_n', \tag{3}$$

where

$$D_n'=\left|\binom{n+1+r}{2s}\right| \qquad (r, s=0, 1, \cdots, n). \tag{4}$$

Indeed if we put

$$E_n=\left|\binom{r}{s}\right| \qquad (r, s=0, 1, \cdots, n),$$

then, by the multiplication formula for determinants,

$$D_nE_n = |c_{rs}|, \qquad c_{rs} = \sum_{k=0}^{n} \binom{n+1}{2s-k}\binom{r}{k} = \binom{n+1+r}{2s}.$$

Since $E_n = 1$, (3) follows at once.

Generalizing (4), we let

$$\Delta_n(x) = \left|\binom{x+r}{2s}\right| \qquad (r, s = 0, 1, \cdots, n). \tag{5}$$

If we subtract the $(n-1)$th row from the nth, the $(n-2)$th from the $(n-1)$th, and so on, we find that $\Delta_n(x) = \Delta_n'(x)$, where

$$\Delta_n'(x) = \left|\binom{x+r}{2s+1}\right| \qquad (r, s = 0, 1, \cdots, n-1).$$

Since

$$\binom{x+r}{2s+1} = \frac{x+r}{2s+1}\binom{x+r-1}{2s},$$

we may multiply each element of the sth column by $2s+1$ and then divide each element of the rth row by $x+r$, and then we have

$$\Delta_n'(x) = \frac{x(x+1)\cdots(x+n-1)}{1\cdot 3\cdots(2n-1)}\left|\binom{x+r-1}{2s}\right|,$$

where $r, s = 0, 1, \cdots, n-1$. Using (5) this gives the recursion

$$\Delta_n(x) = \frac{x(x+1)\cdots(x+n-1)}{1\cdot 3\cdots(2n-1)}\,\Delta_{n-1}(x-1). \tag{6}$$

Since $\Delta_0(x) = 1$, repeated application of (6) yields the explicit formula

$$\Delta_n(x) = \frac{(x)_n(x-1)_{n-1}\cdots(x-n+1)_1}{1^n\, 3^{n-1}\cdot 5^{n-2}\cdots(2n-1)}, \tag{7}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$; this result may be exhibited in the following form. For n odd, $n = 2m-1$,

$$C\Delta_n(x) = x^m \prod_{k=1}^{m-1} \{x^2 - (2k-1)^2\}^{m-k}\{x^2 - (2k)\}^{m-k};$$

for n even, $n = 2m$,

$$C\Delta_n(x) = \prod_{k=0}^{m-1} \{x^2 - (2k)^2\}^{m-k}\{x^2 - (2k+1)^2\}^{m-k},$$

where in either case $C = 1^n 3^{n-1} 5^{n-2}\cdots(2n-1)$.

To get (2) rapidly we remark that

$$D_n = \Delta_n(n+1). \tag{8}$$

Thus substitution of (8) in (6) leads to

$$D_n = \frac{(n+1)(n+2)\cdots 2n}{1\cdot 3\cdots(2n-1)} D_{n-1} = \frac{(2n!)}{n!} \frac{2\cdot 4\cdots 2n}{(2n)!} D_{n-1} = 2^n D_n.$$

Since $D_0 = 1$, (2) follows at once.

We note also that (7) implies

$$\left|\binom{n}{2s-r}\right| = \left|\binom{n+r}{2s}\right| = 2^{n(n-1)/2}, \tag{9}$$

$$\left|\binom{k}{2s-r}\right| = \left|\binom{k+r}{2s}\right| = 0 \qquad (-n+1 \leqq k \leqq n-1), \tag{10}$$

where in each case $r, s = 0, 1, \cdots, n$.

If we multiply $\Delta_n(k)$ by the Vandermonde determinant, $V_n = |a_s^{2r}|$ $(r, s = 0, \cdots, n)$, we get for $0 \leqq k \leqq n+1$

$$\Delta_n(k) V_n = 2^{-n-1} \left| (1+a_s)^{k+r} + (1-a_s)^{k+r} \right| \qquad (r, s = 0, \cdots, n).$$

Consequently, it follows from (8), (9), and (10) that

$$\left| (1+a_s)^{k+r} + (1-a_s)^{k+r} \right| = \begin{cases} 2^{n+1+k(k-1)/2} \prod_{n \geqq i > j \geqq 0} (a_i^2 - a_j^2) & (k = n, n+1) \\ 0 & (0 \leqq k \leqq n-1). \end{cases}$$

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A NOTE ON AUTOMETRIZED BOOLEAN ALGEBRAS

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An "autometrized Boolean algebra" has been defined by Ellis [1] as a Boolean algebra B in which the "distance" of two elements x and y is taken to be their symmetric difference (denoted $d(x, y) = x \oplus y$); this "distance" function satisfies formally the axioms for a metric. In this note we shall give simple and natural proofs for the theorems of [2], concerning the "motions" of B. (A motion is a one to one map of B onto itself preserving "distance.")

We first prove a lemma, from which the other results follow immediately. (The notation is that of [1] and [2].)

LEMMA. *The motions of B are the functions $f(x)=x\oplus a$, where a may be any (constant) element of B.*

Proof. Let f be any motion. Then by definition, $d(x, y)=d(f(x), f(y))$ or $x\oplus y=f(x)\oplus f(y)$. Putting $y=0$ we get $f(x)=x\oplus a$, where we have let $a=f(0)$. Conversely, for each $a\in B$, $f(x)=x\oplus a$ is a motion, since $x\oplus y=x\oplus y\oplus(a\oplus a)=(x\oplus a)\oplus(y\oplus a)=f(x)\oplus f(y)$. We now apply this result to prove:

THEOREM 1 of [2]. *If f is a motion of B, $f(x')=f'(x)$.*

Proof. Let $f(x)=x\oplus a$. Then $f'(x)=(x\oplus a)'=1\oplus x\oplus a=x'\oplus a=f(x')$.

THEOREM 2 of [2]. *The group of motions of B is isomorphic with the additive group $G(B)$ of the associated Boolean ring.*

Proof. Consider the correspondence $c: f(x)=x\oplus a\leftrightarrow a\in G(B)$. The product of two motions f and g is defined as the composition $f(g)$. But if $g(x)=x\oplus b\leftrightarrow b$, then $f(g(x))=x\oplus b\oplus a\leftrightarrow b\oplus a$. Hence c is a homomorphism; since (also from the lemma) c is one to one and onto, Theorem 2 is proved.

The use of this lemma also allows very simple proofs to be given for Theorems 3.1 and 3.2 of [1].

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NOTES ON MATRIX THEORY—IX

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1. Introduction. In a recent note [1], we showed that the inequality

$$|\lambda A+(1-\lambda)B| \geqq |A|^{\lambda}|B|^{1-\lambda}, \tag{1}$$

valid for positive definite matrices A and B, for $0\leqq\lambda\leqq 1$, was a simple consequence of Hölder's inequality and the identity

$$\int_0^{\infty} e^{-(x,Cx)}\prod_i dx_i=\sqrt{\pi}^n/|C|^{1/2}, \tag{2}$$

for C a positive definite matrix of order n.

In this note we wish to use a more recondite identity, a generalization of an integral of Ingham and Siegel, due to A. Selberg, to derive an extensive generalization of (1), namely,

THEOREM. *Let A and B be two positive definite matrices of order n, and let $C=\lambda A+(1-\lambda)B$, for $0\leqq\lambda\leqq 1$. For each $j=1, 2, \cdots, n$, let $A^{(j)}$ denote the principal submatrix of A obtained by deleting the first $(j-1)$ rows and columns, (in particular, $A^{(1)}=A$). Let $B^{(j)}$, $C^{(j)}$, have similar meanings. If $k_1, k_2, \cdots, k_n$ are*

n real numbers such that

$$\sum_{i=1}^{j} k_i \geqq 0, \qquad j = 1, 2, \cdots, n, \tag{3}$$

then

$$\prod_{j=1}^{n} |C^{(j)}|^{k_j} \geqq \prod_{j=1}^{n} |A^{(j)}|^{\lambda k_j} |B^{(j)}|^{(1-\lambda)k_j}. \tag{4}$$

The above sharp form of the inequality is due to a referee. We shall first present his proof below, and then the proof of a particular case, derived from the identity mentioned above.

2. Proof of theorem. According to Bergström's inequality [2], or a minimum theorem due to Fan [4], we have for $j=1, 2, \cdots, n-1$,

$$\begin{aligned} \frac{|C^{(j)}|}{|C^{(j+1)}|} &\geqq \frac{|\lambda A^{(j)}|}{|\lambda A^{(j+1)}|} + \frac{|(1-\lambda)B^{(j)}|}{|(1-\lambda)B^{(j+1)}|} = \lambda \frac{|A^{(j)}|}{|A^{(j+1)}|} + (1-\lambda)\frac{|B^{(j)}|}{|B^{(j+1)}|} \\ &\geqq \left(\frac{|A^{(j)}|}{|A^{(j+1)}|}\right)^{\lambda}\left(\frac{|B^{(j)}|}{|B^{(j+1)}|}\right)^{(1-\lambda)}. \end{aligned} \tag{1}$$

The desired inequality follows upon writing

$$\begin{aligned} \prod_{j=1}^{n} |C^{(j)}|^{k_j} = \left(\frac{|C^{(1)}|}{|C^{(2)}|}\right)^{k_1}\left(\frac{|C^{(2)}|}{|C^{(3)}|}\right)^{k_1+k_2} \\ \cdots \left(\frac{|C^{(n-1)}|}{|C^{(n)}|}\right)^{k_1+k_2+\cdots+k_{n-1}} |C^{(n)}|^{\sum_{i=1}^{n} k_i}, \end{aligned} \tag{2}$$

and using the condition that $\sum_{i=1}^{n} k_i \geqq 0$, together with the inequality above.

3. Partial Proof. It was shown by Siegel [6, p. 44] that the following generalization of the gamma function integral exists:

$$\int_{X>0} e^{-tr(XY)} |X|^{s-(n-1)/2} \prod_{i \leqq j} dx_{ij} = a_n |Y|^{-s}. \tag{1}$$

Here X and Y are symmetric matrices of order n, with Y positive definite, and the integration is extended over the region of x_{ij}-space in which X is positive definite.

The constant a_n is given by

$$a_n = \pi^{n(n-1)/4}\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)\cdots\Gamma\left(s-\left(\frac{n-1}{2}\right)\right). \tag{2}$$

The integral converges for Re $(s) > (n-1/2)$, and equals the right-hand side.

It was pointed out to the author by A. Selberg that an extension of Siegel's integral exists, namely,

$$(3)\quad \int_{X>0} e^{-tr(XY)} \mid X^{(1)} \mid^{\sum_{i=1}^{n} k_i-(n+1/2)} \mid X^{(2)} \mid^{-k_1} \cdots \mid X^{(n)} \mid^{-k_{n-1}} \prod dx_{ij}$$
$$= b_n \mid Y_n \mid^{-k_n} \mid Y_{n-1} \mid^{-k_{n-1}} \cdots \mid Y_1 \mid^{-k_1},$$

where $X^{(i)}$ is as above, $Y_j=(y_{ij})$, $1 \leqq i, j \leqq k$, and

$$(4)\quad b_n = \pi^{n(n-1)/4}\Gamma(k_n)\Gamma\left(k_n + k_{n-1} - \frac{1}{2}\right)\cdots\Gamma\left(\sum_{i=1}^{n} k_i - \left(\frac{n+1}{2}\right)\right).$$

The integral exists and has the stated value provided that each of the expressions

$$k_n,\ k_n + k_{n+1} - \frac{1}{2},\ \cdots,\ \sum_{i=1}^{n} k_i - \left(\frac{n+1}{2}\right),$$

is positive. Once we have a representation for $\mid Y_n \mid^{-k_n} \mid Y_{n-1} \mid^{-k_{n-1}} \cdots \mid Y_1 \mid^{-k_1} = \psi(Y)$ in the form

$$(5)\quad \psi(Y) = \int_{X>0} \phi(X) e^{-tr(XY)} \prod_{i \leqq j} dx_{ij}$$

with $\phi \geqq 0$, the proof proceeds as in [1].

A proof of (3) and an analogous extension of an integral of Ingham [5], equivalent to Siegel's may be found in [3], together with some applications.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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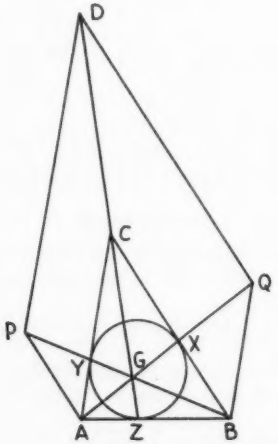
NOTE ON THE GERGONNE POINT OF A TRIANGLE

LAURA GUGGENBUHL, Hunter College, New York

The Gergonne point of a triangle is defined as the point of intersection of the lines from the vertices to the points of contact of the inscribed circle.

An interesting though rarely quoted problem of Joseph Diez Gergonne says [1]: "If through the vertices A and B of a triangle ACB two lines AP, BQ of arbitrary length are drawn in the direction of C, AP parallel to BC, BQ parallel to AC, and if the lines PD and QD are drawn respectively parallel to BQ and AP, meeting in D, then the lines AQ, BP, and CD are concurrent."

The theorem is proved by analytic geometry, with the origin at the vertex C of the triangle, the X-axis on the side CA, and the Y-axis on the side CB.



In passing one may note that if AP and BQ are both taken equal to AB, the point of intersection is the incenter of the triangle; if AP is taken equal to BC, and BQ equal to AC, the point is the median point; and if P is taken as the point at which BY cuts the parallel through A, and Q the point at which AX cuts the parallel through B, where X and Y are points of contact of the incircle, the point of concurrence is obviously the Gergonne point of the triangle.

A curious commentary regarding the origin of the name of the point is suggested by the above problem. The Gergonne point is usually associated with the Nagel point, namely the point of intersection of the lines from the vertices

of a triangle to the internal points of contact of the three escribed circles. In this connection, a reader is frequently referred to Volume 19 of *Nouvelles Annales de Mathématiques* [2]. However, in this source the points, though nameless, are merely included in a list of statements correctly attributed to Nagel [3] and no mention is made of Gergonne.

Nowhere in Gergonne's publications has the author found any reference to the point. In fact, Gergonne did relatively little work in that part of modern geometry which is connected with the notable points and lines of a triangle; and in truth, that part of modern geometry did not really come into its own until some years after Gergonne died. The name is secure in mathematical literature by 1883 [4], but its origin is indeed obscure. It seems certain that the name was given to the point as a memorial, some time after Gergonne's death in 1859. Just why Gergonne was memorialized by this point rather than by any one of a score of others, is truly a provocative question. Perhaps the answer lies in some student's cherished note book; in a budget of innocent errors; or perhaps in some even more likely source.

The author wishes to thank Professor Altschiller Court; M. Goormaghtigh; the referee of this article, Professor Struik; and Professor J. J. Burckhardt of the University of Zurich for several helpful suggestions. Among other things, they have pointed out that the theorem, concerning the lines which join the vertices of a triangle to the points of contact of the incircle, appeared as early as 1678 (on page 38) of Ceva's book, *De Lineis rectis se invicem secantibus statica constructio.*

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THE IMAGINARY NUMBER PROBLEM

A. W. McGaughey, Bradley University

Too many students of college algebra have the preconceived idea that the so called "imaginary" numbers are imaginary, hence see little value in studying them. Even though the instructor mentions that functions of complex variables are very useful in determining constant potential surfaces and streamlines of flow as well as giving other examples of their uses in advanced mathematics, the student is seldom convinced that he is learning about something of a practical nature, or even knows what is being talked about.

I have been unable to document it, but I believe that, in one of Wentworth's algebra texts which was published about the turn of the century, one can find

the following problems, at least three very similar ones, which should help the algebra student to appreciate the value of complex numbers:

1. Two clocks were striking the hour. It was observed that the difference in their number of strokes was six and also that the number of strokes which one made was twice the square of the number of strokes which the other made. How many strokes did each clock make?

2. On a rather chilly morning a man read both his and his neighbor's thermometer. He noticed that one read six degrees higher than the other and that one reading was twice the square of the other reading. What were the temperature readings?

3. Two men left a certain place and walked in straight lines to their destinations which were six miles apart. If the distance which one of them walked was twice the square of the distance which the other man walked, how far did each man walk?

Each of the problems leads to the two sets of equations:

$$y = x + 6, \quad y = 2x^2 \quad \text{and} \quad y = x + 6, \quad x = 2y^2;$$

which have the solutions:

$$x = 2, \quad y = 8; \qquad x = -3/2, \quad y = 9/2;$$
$$x = (-23 + i\sqrt{47})/4, \qquad y = (1 + i\sqrt{47})/4;$$
$$x = (-23 - i\sqrt{47})/4, \qquad y = (1 - i\sqrt{47})/4.$$

Only the first pair of values satisfy the first problem, the first two pairs satisfy the second problem; whereas, all four pairs can be interpreted to satisfy the third problem. For this interpretation we can use the values of x and y as the coordinates in the complex plane of the destinations of the men, assuming they left the origin. We find that they are six miles apart and that one man walked six miles, and the other man walked the square root of three miles.

In 3, the distances are x and $2x^2$, $3/2 \leq x \leq 2$, at angle $\cos^{-1}(x + 1/4x - 9/x^3)$. Ed.

A SIMPLE PROBLEM IN CYLINDRICAL COORDINATES

F. B. HILDEBRAND, Massachusetts Institute of Technology

An elementary calculus class is asked to employ double integration, using circular cylindrical coordinates, for the determination of the volume bounded above by a nappe of the cone $x^2+y^2=z^2$, below by the plane $z=0$, and laterally by the cylinder $x^2+y^2=x$. Many of the students set up the integral as

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^2 dr d\theta \quad \text{or} \quad 2\int_0^{\pi/2} \int_0^{\cos\theta} r^2 dr d\theta$$

and obtain a numerical answer which checks that given in the back of the book. Others, however, are led to the formulation

$$V = \int_0^{\pi} \int_0^{\cos\theta} r^2 dr d\theta,$$

and are less fortunate in the outcome of their calculation.

When the members of the class discuss this formulation, one student points out that here the variable r is not always positive, and hence that the equation of the nappe of the cone must be taken as $z=|r|$, so that, if the range $(0, \pi)$ in θ is insisted upon, the integrand should be written as $|r|r$. He proposes the formulation

$$V = \int_0^{\pi/2}\int_0^{\cos\theta} r^2 dr d\theta - \int_{\pi/2}^{\pi}\int_0^{\cos\theta} r^2 dr d\theta.$$

Calculation yields the correct result and all is well until another student insists that the element of plane area should be taken as $|r|\,dr d\theta$, so that the integrand truly should be written as $|r|^2=r^2$ and the two errors appear to annul each other, leading again to the formulation under criticism.

While the remaining source of error eventually is discovered, to the satisfaction of the instructor, the varied reactions of the students afterward cause the instructor to wonder whether (1) the same discussion should be instigated in other classes or (2) such discussions should be averted in the future by a categorical statement that the range in θ should be so chosen that r is non-negative (presumably with a remark that the treatment of cases in which this is not possible is "beyond the scope of the course") or (3) there is a preferable third alternative.

A CURIOUS FORMULA FOR DISTANCE*

RAYMOND REDHEFFER, University of California, Los Angeles

The following minimax principle gives an interesting exercise in differential calculus:

THEOREM I. *Let R be a convex region in (x, y, z) space containing the points P and Q. Then the distance from P to Q is*

$$d(P, Q) = \sup_f \inf_{(x,y,z)} \frac{f(P) - f(Q)}{\sqrt{f_x^2 + f_y^2 + f_z^2}},$$

where the sup *is taken over all non-constant functions differentiable in R. It does not matter whether the* inf *is taken over all $(x, y, z)\in R$, or only over (x, y, z) on the line segment joining P and Q.*

For proof we use the mean-value theorem to obtain

$$f(P) - f(Q) = (x_1 - x_0)f_x + (y_1 - y_0)f_y + (z_1 - z_0)f_z,$$

where $P=(x_1, y_1, z_1)$, $Q=(x_0, y_0, z_0)$. By the Schwarz inequality we have

* This paper was prepared under the sponsorship of the Office of Naval Research and the Office of Ordnance Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$|f(P)-f(Q)| \leqq H\, d(P, Q)$, where $H=\sqrt{f_x^2+f_y^2+f_z^2}$. Hence at some point of the line PQ, $d(P, Q) \geqq |f(P)-f(Q)|/H$ so that surely $d(P, Q) \geqq \inf\, [f(P)-f(Q)]/H$.

On the other hand, the choice

$$f(x, y, z) = (x_1 - x_0)x + (y_1 - y_0)y + (z_1 - z_0)z$$

yields $[f(P)-f(Q)]/H \equiv d(P, Q)$ as we see by a short calculation. This establishes the theorem.

The result may be applied in various ways, of which the following serves as an illustration:

THEOREM II. *Let the region R be a sphere of radius r, and let M be a positive constant. Then*

$$\inf_f \sup_{(x,y,z)} (f_x^2 + f_y^2 + f_z^2) = (M/r)^2,$$

where the inf *is taken over all differentiable functions f in the closed sphere which satisfy $f \geqq M$ on the boundary and $f=0$ at the center. It does not matter whether we require $f \geqq M$ at one boundary point or $f \geqq M$ at all boundary points.*

With the origin at the center, Theorem I asserts that $\inf\, [f(P)-f(0)]/H \leqq r$ if we choose P on the boundary, and hence $\sup H^2 \geqq (M/r)^2$. On the other hand the choice $f^2=(x^2+y^2+z^2)M^2/r^2$ would give equality, as we see from $ff_x = (M^2/r^2)x$ and from symmetry. This function is not admissible, but is easily approximated by one which is, and the desired result follows. (In two dimensions the reader may imagine a cone with a rounded tip at the origin. An analytical discussion is readily given, but it seems idle to belabor the matter.)

THE FUNDAMENTAL THEOREM OF ARITHMETIC

ROY DUBISCH, Fresno State College

When the student first meets the fundamental theorem of arithmetic he is likely to be puzzled by the "obviousness" of it and, at this level, it does little good to talk about the failure of the theorem for algebraic integers.

The following simple example, then, may be of value. We consider the set of rational numbers, m/n, where m and n are positive integers and $(m, n)=1$. A rational prime is defined as a rational number m/n such that m is an integral prime and n is either an integral prime or 1. Thus it is seen that (1) all integral primes are rational primes; and (2) the definition of an integral prime as an integer p such that $p \neq 1$ and $p=ab$ implies either a or b equal to 1 holds for rational primes. (Note that statements like $(3/5)=(3/10)(2/1)$ are not permissible, since $(3/10)(2/1)=(6/10)$ and $2 \neq 1$.) But, however, the fundamental theorem of arithmetic does not hold for rational primes since, for example, $(6/35)=(3/7)(2/5)=(3/5)(2/7)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1256. *Proposed by W. B. Andreasen, Lockheed Aircraft Corporation*

Discuss the error involved in the following approximate trisection of a circular arc AB. On chord AB locate C such that $BC=BA/3$ and D such that $CD=7AB/6$. With D as center and DC as radius describe an arc to cut arc AB in the approximate trisection point E.

E 1257. *Proposed by N. A. Court, University of Oklahoma*

(1) The medial triangle of each of the four triangles formed by the sides of a complete quadrilateral (q) taken three at a time is homological to the diagonal trilateral of (q).

(2) The four axes of the four homologies coincide.

E 1258. *Proposed by Aaron Herschfeld, Canisius College*

Prove that a necessary and sufficient condition for the rationality of

$$R = \sqrt[3]{a + \sqrt[3]{a + \cdots}},$$

where a is a positive integer, is that $a=N(N+1)(N+2)$, the product of three consecutive integers. In that case find R.

E 1259. *Proposed by C. D. Olds, San Jose State College*

A building a feet square has a walk $b<a$ feet wide around it. Two persons are on the walk; find the chance that they can see each other.

E 1260. *Proposed by Viktors Linis, University of Ottawa*

Evaluate $\int_0^{\pi/2} \cot\theta \ln \sec\theta d\theta$.

SOLUTIONS

The Red and White Cows

E 1226 [1956, 491]. *Proposed by J. V. Pennington, Houston, Texas*

A rancher bought a white cow, and in the following year a red one. Each succeeding year he duplicated his purchases of the preceding two years, buying the same number of cows, of the same colors and in the same order. Thus, in the third year, he bought a white and then a red cow; in the fourth year, a red,

then a white, and then a red cow; and so on. What was the color of the nth cow?

Solution by T. F. Mulcrone, Loyola University. If we associate with a white cow the number 1 and with a red cow the number 2, then the sequence defined in the problem becomes

$$\{a_n\} = 1; 2; 1, 2; 2, 1, 2; 1, 2, 2, 1, 2; \cdots,$$

whose nth term $a_n = [kn] - [k(n-1)]$, where $k = (\sqrt{5}+1)/2$ and $[x]$ is the greatest integer not exceeding x (see problem 4247 [1948, pp. 588–592]). Thus the nth cow is white or red according as a_n is 1 or 2.

Also solved by J. P. Ballantine, A. R. Hyde, P. G. Kirmser, D. C. B. Marsh, Azriel Rosenfeld, E. D. Schell, and the proposer. Late solutions by Hazel E. Evans, Celestine O'Callaghan, and D. S. Passman.

The proposer gave also the equivalent statement, "The nth cow is red or white according as the fractional part of $n(\sqrt{5}+1)/2$ is or is not less than $(\sqrt{5}-1)/2$."

A Property of e

E 1227 [1956, 491]. *Proposed by R. L. Helmbold, Carnegie Institute of Technology*

Find all values of $a \geqq 1$ such that $a^x \geqq x^a$ holds for all values $x \geqq 0$.

Solution by D. S. Greenstein, Radio Corporation of America, Camden, N. J. The restriction $a \geqq 1$ may be replaced by $a \geqq 0$. Taking logarithms and dividing by the positive quantity ax, we must have $(\ln x)/x \leqq (\ln a)/a$. Since $(\ln x)/x$ attains its maximum uniquely at $x = e$, it follows that a must equal e.

Also solved by Michael Goldberg, Emil Grosswald, B. A. Hausmann, A. S. Hendler, P. B. Johnson, J. B. Johnston, Sidney Kravitz, Joe Lipman, D. C. B. Marsh, J. W. Mettler, J. B. Muskat, C. S. Ogilvy, L. A. Ringenberg, Azriel Rosenfeld, E. D. Schell, Anina Schub, A. V. Sylwester, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by I. M. Isaacs and D. S. Passman.

Editorial Note. Following are some MONTHLY references related to E 1227: [1916, 233–237], [1921, 141–143], [1931, 444–447], [1936, 229–230], E 640 [1945, 278], E 853 [1949, 555], E 1144 [1955, 446]. The problem is dealt with in Euler's *Introductio in Analysin Infinitorum*, II, p. 294. As the proposer remarked, E 1227 settles without computation the old problem, "Which is greater, e^π or π^e?"

An Affine Invariant

E 1228 [1956, 491]. *Proposed by Viktors Linis, University of Ottawa*

Let $n(P)$ be the number of distinct lines L through a point P such that L divides the area of a given triangle in two equal parts. Show that the locus of all points P with $n(P) \geqq 2$ is a region the ratio of whose area to the area of the given triangle is an absolute constant.

Solution by the proposer. The theorem is a consequence of the facts that $n(P)$ and the ratio of areas are affine invariants and that any triangle is affinely equivalent to an equilateral triangle.

For an equilateral triangle the required region is bounded by three arcs of hyperbolas, each of which has two sides of the triangle for asymptotes, and

pairs of which are tangent to one another at the midpoints of the medians. Taking the side of the equilateral triangle as the unit length, the rectangular equation of one of the hyperbolas referred to one side and one vertex of the triangle as x-axis and origin is $y(\sqrt{3}x-y)=3/16$. The area of the region can be obtained by simple integration as $\sqrt{3}\ (\ln 8-2)/16$, and hence the required constant is $c=(\ln 8-2)/4\approx 0.0198$.

Also solved by Michael Goldberg, J. B. Johnston, and M. S. Klamkin.

Editorial Note. For an allied problem see E 774 [1950, 484].

Configurations of Four Coplanar Points

E 1229 [1956, 492]. *Proposed by M. P. Drazin, Trinity College, Cambridge, England*

Given any point O in the plane of a triangle $\Delta\equiv ABC$, let the sides a, b, c of Δ subtend angles A', B', C' at O, and let the distances from O to the vertices of Δ be a', b', c'. Show that the triangle with sides aa', bb', cc' has angles $A'-A$, $B'-B$, $C'-C$, and find the sextic polynomial relation connecting a, b, c, a', b', c'.

Solution by the proposer. (i) Take, for definiteness, the case in which O lies inside Δ, and construct, externally to Δ, triangle PCB directly similar to triangle OAB. Then $\sphericalangle OBP=\sphericalangle ABC$ and $BP/BO=BC/BA$, so that triangle OBP is similar to the given triangle ABC, corresponding sides being in the ratio b'/c. Consequently $\sphericalangle BOP=\sphericalangle BAC$ and $OP=bb'/c$, while also $PC=aa'/c$. Thus triangle OCP has sides aa'/c, bb'/c, c' and angle $A'-A$ at O, whence the first assertion follows by symmetry.

As an application, we deduce, on taking $a'=b'=c'=R$, the well known result that, given any circle ABC having its center O on the same side of BC as A, then the angle subtended by BC at O is twice that subtended at A.

(ii) Since $A'+B'+C'=2\pi$, we have

$$1+2\cos A'\cos B'\cos C'=\cos^2 A'+\cos^2 B'+\cos^2 C'.$$

On substituting $(b'^2+c'^2-a^2)/2b'c'$ for $\cos A'$, and similar expressions for $\cos B'$ and $\cos C'$, we obtain

$$4a'^2b'^2c'^2+(b'^2+c'^2-a^2)(c'^2+a'^2-b^2)(a'^2+b'^2-c^2)$$
$$=a'^2(b'^2+c'^2-a^2)^2+b'^2(c'^2+a'^2-b^2)^2+c'^2(a'^2+b'^2-c^2)^2,$$

or, after a little simplification,

$$a^2b^2c^2+\Sigma\, a^4a'^2+\Sigma a^2(a'^2-b'^2)(a'^2-c'^2)=\Sigma b^2c^2(b'^2+c'^2)+\Sigma(b^2+c^2)b'^2c'^2.$$

A more symmetric form of this can be obtained by treating O, A, B, C on equal footing and replacing a, b, c, a', b', c' by $d_{23}, d_{31}, d_{12}, d_{01}, d_{02}, d_{03}$.

Partially solved by D. C. B. Marsh, who obtained the sextic relation.

The *n*th Integer Prime to 30

E 1230 [1956, 492]. *Proposed by Vassili Daiev, Sea Cliff, Long Island*

Find the nth term of the sequence of ordered positive integers prime to 30.

Solution by J. B. Muskat, Allston, Mass. Since $\phi(30)=8$, let $n=8q+r$, $0 \leqq r<8$. Define $f(r)$ as follows: $f(0)=-1$, $f(1)=1$, $f(2)=7$, $f(3)=11$, $f(4)=13$, $f(5)=17$, $f(6)=19$, $f(7)=23$. Then the nth term a_n is $30q+f(r)$.

Also solved by D. A. Breault and Underwood Dudley (jointly), Leonard Carlitz, Monte Dernham, Hazel E. Evans, A. R. Hyde, P. B. Johnson, M. S. Klamkin, Joe Lipman, D. C. B. Marsh, G. E. Meador, Herbert Nadler, C. S. Ogilvy, L. A. Ringenberg, Azriel Rosenfeld, E. D. Schell, and the proposer. Some of these solutions were not entirely correct. Late solution by D. S. Passman.

The answer can be given in many forms. Thus Marsh gave the formula

$$a_n = 30[n/8] - 1 + 2r + 4[r/2] + 2[r/3] - 4[r/4] + 2[r/5] - 6[r/6] + 2[r/7],$$

where r is the remainder after n is divided by 8, and $[x]$ is the greatest integer not exceeding x; Ogilvy gave the formula

$$a_n = 2[(5n + 1 \pm 1)/4] + 2[(5n + 1 \pm 1)/8] - 1,$$

where the plus or minus is chosen according as the residue of n modulo 8 does not or does exceed 3; Ringenberg gave the formula

$$a_n = 30p + 15 + (-1)^q(k^2 - k + 2),$$

where

$$p = [(n - 1)/8], \qquad q = [(n + 3)/4], \qquad k = |\, n - 8p - 9/2\,| + 1/2;$$

and the proposer gave the formula

$$\begin{aligned} 4a_n = 15n - \{&11\langle (n-1)/8\rangle + 2\langle (n-2)/8\rangle + \langle (n-3)/8\rangle \\ &+ 8\langle (n-4)/8\rangle + 7\langle (n-5)/8\rangle + 14\langle (n-6)/8\rangle \\ &+ 13\langle (n-7)/8\rangle + 4\langle (n-8)/8\rangle\}, \end{aligned}$$

where $\langle x\rangle$ is 1 or 0 according as x is or is not an integer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4270 [1957, 49]. *Correction.* The number should be 4720.

4728. *Proposed by R. P. Boas, Jr., Northwestern University*

A. M. Rodov has propounded a proof that if $f(x)$ is continuous and the first of the following integrals converges, then the second diverges

$$\int_1^\infty f(x)dx, \qquad \int_1^\infty x^{-2}\{f(x)\}^{-1}dx.$$

(a) Construct a counter-example. (b) More generally, show that if $g(x)$ and $\phi(x)$ are positive and $\int_1^\infty \phi(x)dx$ diverges, then at least one of

$$\int_1^\infty \phi(x)g(x)dx \quad \text{and} \quad \int_1^\infty \{\phi(x)/g(x)\}dx$$

diverges.

4729. *Proposed by Paul Erdös, The University, Birmingham, England*

Let b_k, $k=1, 2, \cdots$, be any sequence of non-negative integers such that $\limsup b_k^{1/k}<2$. Assume further that $\sum_{k=1}^n b_k\to\infty$, $\liminf \sum_{k=1}^n b_k/n=0$. Prove that $\sum_{k=1}^\infty b_k/2^k$ is irrational.

4730. *Proposed by E. J. F. Primrose, University College, Leicester, England*

If a finite set of points in complex 3-dimensional space has the property that the line joining any two points of the set passes through a third point of the set, must all points of the set be coplanar?

4731. *Proposed by D. S. Stoller, Los Angeles, Calif.*

Consider an imperfect sorting process acting on a very large number of items, each of which belongs in one and only one of k bins. Let Q_i represent the probability that if an item belongs in bin i, it is sorted into bin i, and if it does not belong in bin i, it is sorted into some other unspecified bin. Find Q, the probability that an item is sorted into the correct bin.

4732. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Show that, in the ring of polynomials with integer coefficients, $(J(x))$, there are ideals which require arbitrarily many generators.

SOLUTIONS

Summation of Inverse Tangents

4678 [1956, 191]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Hartford (Conn.) Graduate Center*

Evaluate $\tan^{-1} 1 - \tan^{-1}(1/3) + \tan^{-1}(1/5) - \cdots$.

I. *Solution by Kovina Milosevich, Mathematical Institute, Skopje, Yugoslavia.* Starting with the known expansion

$$\frac{\pi}{4\cosh(\pi x/2)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n-1)^2 + x^2}$$

and integrating, we obtain $\tan^{-1} e^{\pi x/2} - \pi/4 = \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1}\{x/(2n-1)\}$. The required result is the special case which results upon letting $x \to 1$ and employing Abel's theorem. This gives

$$\tan^{-1} e^{\pi/2} - \pi/4 = \tan^{-1}(\tanh(\pi/4)) = \tfrac{1}{2}\tan^{-1}(\sinh(\pi/2)) = \tfrac{1}{2}gd(\pi/2).$$

II. *Solution by M. A. Rashid, Panjab University, Lahore, Pakistan.* De Moivre's theorem shows that, if

$$(1 + ib_1/a_1)(1 + ib_2/a_2)\cdots \to A + iB,$$

then

$$\tan^{-1}(b_1/a_1) + \tan^{-1}(b_2/a_2) + \cdots \to \tan^{-1}(B/A).$$

We have, therefore, $(1+ix)(1-ix/3)(1+ix/5)\cdots$

$$= \frac{Z}{\pi/4} \cdot \frac{1 - Z/\pi}{1 - (\pi/4)/\pi} \cdot \frac{1 + Z/\pi}{1 + (\pi/4)/\pi} \cdots,$$

$$= \frac{(Z)(1 - Z^2/\pi^2)(1 - Z^2/2^2\pi^2)\cdots}{(\pi/4)(1 - (\pi/4)^2/\pi^2)(1 - (\pi/4)^2/2^2\pi^2)\cdots},$$

$$= \frac{\sin Z}{\sin(\pi/4)} = \frac{\sin\{(1+ix)\pi/4\}}{\sin(\pi/4)} = \cosh(\pi x/4) + i\sinh(\pi x/4),$$

where we have put Z for $(1+ix)\pi/4$. Hence

$$\tan^{-1} x - \tan^{-1}(x/3) + \tan^{-1}(x/5) - \cdots = \tan^{-1}(\tanh(\pi x/4)),$$

with the special case $x = 1$ for the proposed series.

Also solved by A. D. Anderson, Leon Bankoff, Leonard Carlitz, A. E. Danese, R. B. Deal,

H. E. Fettis, G. B. Findley, N. J. Fine, H. E. Goheen, Peter Henrici, L. I. Lowell, Justin MacCarthy, A. J. Macintyre, M. Morduchow, F. D. Parker, Paul Payette, D. A. Robinson, D. C. Russell, Daniel Shanks, Leon Steinberg, Ernst Trost, Chih-yi Wang, R. E. Wild, J. E. Wilkins, Jr., Louise A. Wolf, David Zeitlin, and the proposer.

Editorial Note. Several readers pointed out that the formula was stated by Ramanujan, *Collected Papers*, p. 42. It also appears as exercise 16, p. 370, in Hobson, *Treatise on Plane Trigonometry*, 6th ed., 1925.

Ten Concyclic Orthopoles

4679 [1956, 191]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

Solution by Chih-yi Wang, University of Minnesota. We make use of the following known

THEOREM. *If a line meets the circumcircle of a triangle, the Simson lines of the points of intersection with the circle meet in the orthopole of the line for the triangle.* (See Court, *College Geometry*, 2nd ed., p. 289.)

Let the circumcircle $A_1A_2A_3A_4A_5$ be the unit circle, and the coordinates of A_i be $(\cos\theta_i, \sin\theta_i)$, $i=1, 2, \cdots, 5$. For definiteness let us find the coordinates of Ω_{12}. The equations of the Simson lines of A_1 and of A_2 are given by

$$y - \tfrac{1}{2}(\sin\theta_j + \sin\theta_3 + \sin\theta_4) = \tfrac{1}{2}\sin(\theta_3 + \theta_4 - \theta_j)$$

$$= \tan\tfrac{1}{2}(\theta_3 + \theta_4 + \theta_5 - \theta_j)\left[x - \tfrac{1}{2}(\cos\theta_3 + \cos\theta_4 + \cos\theta_j) + \tfrac{1}{2}\cos(\theta_3 + \theta_4 - \theta_j)\right],$$

for $j=1, 2$. By solving the simultaneous equations we obtain

$$\Omega_{12} = (\alpha + \tfrac{1}{2}\cos(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2),\ \beta + \tfrac{1}{2}\sin(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2)),$$

where $\alpha = \frac{1}{2}\sum\cos\theta_k$, $\beta = \frac{1}{2}\sum\sin\theta_k$, $k=1, 2, \cdots, 5$. Since α and β are symmetric functions, by interchanging the subscripts we see readily that the ten points Ω_{ij} all lie on the circle of radius $\frac{1}{2}$ with center (α, β).

Also solved by J. W. Clawson, R. Goormaghtigh, O. J. Ramler, Sister M. Stephanie, and the proposer.

Editorial Note. Goormaghtigh gave this theorem in *Mathesis*, 1939, p. 312. Ramler gives an extension to the cyclic heptagon. If Ω_{ijk} denotes the Kantor point of a triangle $A_iA_jA_k$ with respect to the quadrangle formed by the remaining four vertices, then the thirty-five points Ω_{ijk} all lie on a circle one-half as large as the circumcircle of the heptagon.

Another Generalization of Clairaut's Differential Equation

4680 [1956, 191]. *Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.*

Solve the following generalization of Clairaut's differential equation

$$y - xy' + \frac{x^2y''}{2!} - \cdots + (-1)^{n-1}\frac{x^{n-1}y^{(n-1)}}{(n-1)!} + (-1)^n\frac{x^nF(y^{(n)})}{n!} = G(y^{(n)}).$$

Solution by the proposer. After differentiating the given equation and replacing $y^{(n)}$ by r, we can rewrite it as

$$dx^n/dr - x^nF'(r)H(r) = (-1)^{n-1}G'(r)n!H(r), \quad \text{where} \quad H(r) = (r - F(r))^{-1}.$$

The standard solution of this first order linear equation is

$$x^n = \exp\left\{\int F'(r)H(r)dr\right\}\int (-1)^{n-1}G'(r)n!H(r)\exp\left\{-\int F'(r)H(r)\,dr\right\}dr,$$

or $x=\phi(r)$. Now $(d^ny)/(dx^n)=r$, whence

$$y = \int\cdots\int r(dx)^n = \int \phi'(r)\int \phi'(r)\cdots\int r\phi'(r)(dr)^n.$$

This last equation and the equation $x=\phi(r)$ constitute the parametric form of the solution.

It is to be noted that there is no singular solution unless $r=F(r)$ in which case the equation reduces to that treated by Witty, this MONTHLY, 1952, pp. 100–102. See also the proposer's note, this MONTHLY, 1953, pp. 97–99.

Also solved by Muneer A. Rashid and Chih-yi Wang.

A Summation Problem

4682 [1956, 191]. *Proposed by R. C. Lyness, Preston, England*

(a) Prove that when the series

$$1 + \sum_{r=1}^{\infty}\binom{r\alpha}{r-1}\frac{x^r}{r}$$

is convergent, its sum, y, satisfies $y=1+xy^\alpha$.

(b) Prove also that

$$\sum_{r=1}^{\infty}\binom{r\alpha+\beta-1}{r-1}\frac{x^r}{r} = \frac{y^\beta - 1}{\beta}.$$

I. *Solution by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.* Case (a) is a special case of (b) which, in turn, is an application of Lagrange's reversion formula (See Bromwich, *Infinite Series*, p. 158): If $y=xf(y)$, then $g(y)=\sum_0^\infty p_ny^n$, where np_n is the coefficient of y^{-1} in the expansion of $g'(y)/x^n$.

Here $g(y)=(y^\beta-1)/\beta$, and $f(y)=y^{\alpha+1}/(y-1)$. It follows that the coefficient of y^{-1} in the expansion of $y^{\beta-1}y^{\alpha n}(y-1)^{-n}$ is

$$(-1)^n\binom{n-1-\alpha n-\beta}{n-1} = \binom{n\alpha+\beta-1}{n-1},$$

by the binomial theorem. Thus

$$\frac{y^\beta - 1}{\beta} = \sum_{n=1}^{\infty} \binom{n\alpha + \beta - 1}{n - 1} \frac{x^n}{n}.$$

II. *Solution by Chih-yi Wang, University of Minnesota.* This is a simple and interesting application of relation (7) in Gould's paper, ***Some generalizations of Vandermonde's convolution***, this MONTHLY, 1956, pp. 84–91. Following Gould's notation we have, since $x = (y-1)/y^\alpha$,

$$\sum_{r=1}^{\infty} \binom{r\alpha + \beta - 1}{r - 1} \frac{x^r}{r} = \frac{1}{\beta}\left[\sum_{r=0}^{\infty} A_r(\alpha, \beta)x^r - 1\right] = \frac{1}{\beta}(y^\beta - 1).$$

This is (b) and $\beta = 1$ gives (a). Gould notes that these results are valid if $|x| < |(\alpha-1)^{\alpha-1}/\alpha^\alpha|$.

Also solved by H. W. Gould and Nathaniel Grossman.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Automatic Digital Computers. By M. V. Wilkes, F. R. S., Wiley, New York, 1956, x+305 pages, $7.00.

This book is a good survey of the digital computing field. Being a survey, it will probably not satisfy each specialist in his own line, but it will give him a good understanding of the problems arising in other lines. As the author states, the book will not enable a competent computer-programmer to design automatic digital computers, nor will it enable a computer-designer to become a competent computer-programmer.

The book contains excellent sections on the history of automatic computers, the design of many of the present computers, the philosophy of coding, and the philosophy of computer design. The author goes quite deeply into the electronic circuitry of computers and into coding techniques in order to delineate the still unresolved controveries, such as serial against parallel design and automatic machine compiling of programs against direct coding in machine language.

The typography is clear and the illustrations both profuse and of good quality. One of the most valuable contributions is the comprehensive annotated bibliography.

This book should certainly be in the library of every large-scale computing facility and will be of real interest to all who are associated in any way with the digital computing field.

JOHN E. MAXFIELD
Naval Ordnance Test Station
China Lake, Calif.

Mathematics of Finance. By Robert and Helen Cissell. Houghton Mifflin, Boston, 1956. ix+198+88 pages, $4.50.

The nine chapters in this attractive looking textbook contain the material usually covered in a one-semester elementary course in the mathematics of investment. The chapter headings are: Simple Interest and Bank Discount, Compound Interest and Discount, Ordinary Annuities, Other Annuities Certain, Amortization and Sinking Funds, Bonds, Depreciation, Life Annuities, Life Insurance. There is no review of topics such as logarithms and progressions. The book contains a good selection of problems; answers to the odd-numbered problems are given.

This textbook has several excellent features. The many well-drawn time diagrams should prove especially helpful to the student. The authors point out common errors that students frequently make in setting up equations; these advance warnings should help to prevent the occurrence of some of the usual mistakes.

The various figures and charts in the book serve to stimulate interest in the subject. There are reproductions of notes, drafts, bonds, bond tables, and an F. H. A. amortization schedule; graphs illustrate the premium paid and the protection given under various life insurance plans. Practical bits of information are given in the hope that the student will become a wise borrower and investor; he is informed on topics such as sources of credit and the interest rates charged, the problems involved in financing the purchase of a home, and the wisdom of purchasing various types of life insurance.

One feature will be especially appreciated by students. An index of forty-one formulas is printed inside the back cover. These formulas are classified as to type (simple interest, ordinary annuities, etc.), and opposite each formula there is listed its use, followed by the page of the text on which the formula was introduced.

The tables deserve favorable comment. The fact that they are printed on yellow paper in very clear black ink makes them easy to locate and read. The logarithm tables are six-place tables, and the compound interest and annuity tables are ten-place tables. The arrangement of these ten-place tables is one not usually found in mathematics of finance books. (Cissells' tables were reproduced from the Compound Interest Tables of the Financial Publishing Company). The reviewer found that she was able to locate values much more rapidly in Cissells' tables than she could in the tables that she was accustomed to using.

Adverse criticism of the book is directed at the style of writing of the authors.

Excessive use is made of the first person plural. Explanations are phrased in the terminology that an instructor might use if he were talking quite casually to his class, but not in the terminology that he would use if he knew his words were to be recorded in print. A sentence on page 119 will illustrate this point: "As long as you owe several thousand dollars, you will be paying a lot of interest every year even though you have a relatively low rate."

In their attempts to oversimplify definitions and the presentation of new material, the authors have sometimes made statements which lack clarity and which do not have the precise meaning that was intended. For example, in their chapter on compound interest and discount, the authors have a section headed *Use of Logarithms*. In this section they state the following: "Logarithms can be used to solve problems when compound interest tables are not available or when the time or rate is not in the table. . . . If time is determined by logarithms and simple interest is to be used for the remaining fraction of a period, the amount should be determined for the integral number of periods. Then the simple interest formula can be used to get the additional time needed for this sum to equal the final amount." This is the complete quotation on that particular use of logarithms, and there is given no example which might clarify the authors' intended meaning.

If one is considering this textbook for possible classroom adoption, he should study carefully both its good points and its failings before making a decision.

VIOLET HACHMEISTER LARNEY
State University of New York
College for Teachers, Albany

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL SCIENCE FOUNDATION SUMMER INSTITUTES

The National Science Foundation has announced the following Summer Institutes for college and high school teachers of mathematics. Inquiries about a particular institute should be sent to the Director listed below. Unless otherwise indicated, the Director is located at the same institution as the institute.

University of Colorado, June 24 to July 20: College teachers only. Professor R. E. Langer, University of Wisconsin, Madison.

University of Kansas, June 10 to August 2: High school and college teachers. Professor G. B. Price.

High school teachers only:

Columbia University, Teachers College, July 8 to August 16: Professor H. F. Fehr.

Indiana University, June 17 to August 9: Mrs. Marie S. Wilcox.
Miami University, June 17 to July 26: Professor T. C. Holyoke.
Montana State College, June 22 to August 23: Professor J. W. Hurst.
Polytechnic Institute of Puerto Rico, June 1 to July 27: Professor Mariano Garcia, College of Agriculture and Mechanical Arts, University of Puerto Rico, Mayaguez, P. R.
Teachers College at Oneonta, New York (for Junior High School Teachers), July 1 to August 9: Professor Vera Sanford.
State University of Iowa, June 17 to July 27: Professor L. A. Knowler.
University of Buffalo, July 8 to August 2: Professor Harriet F. Montague.
University of Chicago, June 24 to August 2: Professor A. L. Putnam.
University of Colorado, June 17 to July 27: Professor B. W. Jones.
University of Massachusetts, July 1 to August 16: Professor R. W. Wagner.
University of Notre Dame, June 21 to August 6: Professor A. E. Ross.
University of Wyoming, June 17 to August 9: Professor W. N. Smith.

PERSONAL ITEMS

St. Louis University: Dr. John Riner, Head of Department of Mathematics, St. Peter's College, has been appointed Assistant Professor; Mrs. Virginia H. Kern, McDonnell Aircraft Company, has been appointed Lecturer.

University of Massachusetts: Miss Elaine Cook, University of West Virginia, Miss Betty M. Navratil, Mathematical Analyst, Collins Radio Company, Cedar Rapids, Iowa, and Mr. D. H. Trahan have been appointed Instructors; Mr. R. E. Libera, Student, American International College, Mr. J. A. Pavelcak, Teacher, Hopkins Academy, Hadley, Massachusetts, and Mrs. Doris Stockton, Graduate Student, Brown University, have been appointed part-time Instructors; Dr. D. E. Moser has been promoted to Assistant Professor.

Dr. A. W. Adler has been appointed Instructor at Princeton University.

Dr. H. I. Ansoff, Rand Corporation, has accepted a position as Development Planning Specialist with the Lockheed Aircraft Corporation, Burbank, California.

Mr. J. D. Armstrong, Bolles School, is employed as a mathematician and computer at the Army Ballistic Missile Agency, Redstone Arsenal, Huntsville, Alabama.

Mr. G. E. Barlow, Jr., Assistant County Agent, Gate City, Virginia, has a position as an extension agricultural engineer at Iowa State College.

Dr. L. C. Barrett, University of Utah, has been appointed Associate Professor at Arizona State College.

Mr. H. H. Berry, Senior Research Engineer, Crosley Division, AVCO Manufacturing Company, Cincinnati, Ohio, is Associate Mathematician, Armour Research Foundation, Illinois Institute of Technology.

Miss H. Christine Boyd, Graduate Assistant, University of Mississippi, has been appointed Assistant Professor at Louisiana Polytechnic Institute.

Mr. C. M Braden, Institute of Technology, University of Minnesota, has been appointed Assistant Professor at Macalester College.

Associate Professor J. L. Brenner, State College of Washington, has accepted a position as a mathematician with the Stanford Research Institute.

Associate Professor Eleazer Bromberg, New York University, has been promoted to Professor.

Mr. H. H. Brown, Franklin Institute, has accepted a position as a mathematician on the Technical Staff, Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor H. E. Campbell, Emory University, has been appointed Assistant Professor at Michigan State University.

Dr. W. C. Carter, Raytheon Manufacturing Company, has accepted a position as

Manager, Systems Programming Department, Datamatic Corporation, Newton Highlands, Massachusetts.

Professor Lamberto Cesari, Purdue University, was Visiting Lecturer at the Universities of Sao Paulo and Rio de Janeiro during the summer of 1956.

Mr. E. H. Connell, Consolidated-Vultee Aircraft Corporation, has accepted a position as a senior scientist with Lockheed Aircraft Corporation, Sunnyvale, California.

Mr. C. L. Davis, Mathematician, Curtiss-Wright Corporation, Clifton, New Jersey, has a position as a mathematician programmer with General Motors Research Staff, Detroit, Michigan.

Dr. R. L. Davis, University of Michigan, has been appointed Assistant Professor at the University of Virginia.

Assistant Professor W. E. Deskins, Ohio State University, has been appointed Assistant Professor at Michigan State University.

Professor O. L. Dustheimer, Ohio Northern University, has been appointed Professor at Youngstown University.

Professor H. S. Everett, University of Chicago, is on leave of absence and has been appointed Visiting Professor at Pennsylvania State University.

Mr. C. C. Faith, Michigan State University, has been promoted to Assistant Professor.

Mr. C. K. Fendall, Graduate Student, Reed College, has a position as an associate engineer for Boeing Airplane Company, Seattle, Washington.

Professor Emeritus L. R. Ford was presented with a collection of manuscripts dedicated to him on the occasion of his seventieth birthday, October 28, 1956. The papers were collected by Professors Karl Menger and Gordon Pall. The authors of the papers are: W. R. Abel and L. M. Blumenthal, H. T. Davis, W. L. Duren, Jr., H. J. Ettlinger and J. L. Cornette, G. C. Evans, L. R. Ford, Jr., J. W. Green, R. E. Langer, G. W. Mackey, Karl Menger, Gordon Pall, Tibor Radó, W. T. Scott, H. S. Wall, and G. T. Whyburn.

Assistant Professor Joel Franklin, University of Washington, has accepted a position as senior mathematician with the ElectroData Division, Burroughs Corporation, Pasadena, California.

Miss Joyce B. Friedman, ACF Electronics, has accepted a position as a mathematician with Technical Operations, Inc., Washington, D. C.

Mr. S. W. Golomb, Harvard University, is employed as a senior research mathematician at the Jet Propulsion Laboratory, California Institute of Technology.

Mr. R. M. Gordon, National Cash Register Company, has accepted a position as sales technical specialist with the ElectroData Division, Burroughs Corporation, Pasadena, California.

Mr. L. R. Harper, Jr., has been appointed Instructor at the University of Minnesota.

Assistant Professor Frank Hawthorne, Hofstra College, has been appointed Supervisor of Mathematics, New York State Education Department.

Mr. G. A. Heuer has been appointed Instructor at Concordia College.

Mr. Bernard Jacobson, Michigan State University, has been appointed Assistant Professor at Franklin and Marshall College.

Mr. M. L. Keedy has been appointed Instructor at the University of Nebraska.

Assistant Professor Paolo Lanzano, St. Louis University, is employed by Douglas Aircraft Company, Santa Monica, California.

Mr. C. W. Leininger, University of Texas, has been appointed Assistant Professor at Arlington State College.

Mr. A. J. Leino has accepted a position as a research engineer at Consolidated-Vultee Aircraft Corporation, San Diego, California.

Associate Professor L. H. Loomis, Harvard University, has been promoted to Professor.

Professor T. A. Love, Head, Department of Mathematics, Tennessee Agricultural and Industrial State College, has been appointed Professor and Chairman of the Department of Mathematics, Fisk University.

Mr. R. K. McConnell, Jr., New York University, has been appointed Assistant Professor at the University of Rhode Island.

Dr. D. G. Miller, Research Associate, Chemistry Department, Brookhaven National Laboratory, Upton, New York, has a position as a chemist at the University of California Radiation Laboratory, Livermore, California.

Dr. Josephine Mitchell, General Electric Company, has accepted a position as a research mathematician with Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

Mr. J. T. Morse has accepted a position as a research mathematician with the Carter Oil Company, Tulsa, Oklahoma.

Mr. H. W. Moyer has a position as an associate engineer with Douglas Aircraft Company, Holloman Air Force Base, New Mexico.

Mr. H. L. Newman, Manager, TV Advanced Development, Sylvania Electric Products, Buffalo, New York, is now engineering specialist for Sylvania at Mountain View, California.

Associate Professor M. M. Ohmer, Southwestern Louisiana Institute, has been promoted to Professor.

Dr. J. B. O'Toole, Philco Corporation, has accepted a position as a research physicist with the Hughes Research and Development Laboratories, Culver City, California.

Miss Elaine B. Pavelka, Graduate Student, Northwestern University, is a teacher at Leyden Community High School, Franklin Park, Illinois.

Dr. Mary H. Payne, Columbia University, is employed as a research engineer at Fairchild Guided Missiles Division, Wyandanch, New York.

Dr. C. L. Perry, Jr., U.S. Naval Postgraduate School, has been appointed head of the mathematics group at Stanford Research Institute, Menlo Park, California.

Dr. Ronald Pyke, University of Washington, has been appointed a research associate at Stanford University.

Dr. L. B. Rall, Oregon State College, is employed as a mathematician by the Shell Development Company, Emeryville, California.

Dr. W. P. Reid, Staff Member, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, has been appointed Associate Professor at Michigan State University.

Dr. T. J. Rivlin, New York University, has accepted a position as senior mathematical analyst with Fairchild Engine and Airplane Corporation, Long Island, New York.

Dr. H. L. Rolf, Vanderbilt University, has been appointed Assistant Professor at Baylor University.

Miss Ruth L. Royer, Chico State College, has been promoted to Assistant Professor.

Dr. Stewart Schlesinger, Staff Member, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, is now Manager, Digital Computing Group, Aeronutronic Systems, Glendale, California.

Assistant Professor N. J. Schoonmaker, University of Massachusetts, has been appointed Head, Department of Mathematics, University of Vermont.

Mr. Cecil Schwartz, Senior Power Plant Engineer, Glenn L. Martin Company, Baltimore, Maryland, is a thermodynamics engineer for Marquardt Aircraft Company, Van Nuys, California.

Assistant Professor R. D. Sheffield, University of Mississippi, has a position as a senior nuclear engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. Bernard Sherman, University of California at Los Angeles, has accepted a position as research mathematician with Westinghouse Research Laboratories, Pittsburgh, Pennsylvania.

Associate Professor R. L. Shively, Manchester College, has been appointed Assistant Professor at Western Reserve University.

Dr. Alton H. Smith, University of Southern California, has accepted a position as a mathematician in the Computer Systems Division, Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. Malcolm Smith, Cook Research Laboratories, is now a senior staff member with Motorola, Riverside, California.

Dr. Ella Marth-Snader, District of Columbia Teachers College, has been appointed Professor at Chicago Teachers College.

Mr. W. A. Soper, Jr., Associate Engineer, Westinghouse Electric Corporation, Baltimore, Maryland, has accepted a position as a senior member of the Technical Staff, Airborne Systems Laboratory, Radio Corporation of America, Waltham, Massachusetts.

Dr. R. D. Stalley, Fresno State College, has been appointed Assistant Professor at Oregon State College.

Dr. J. M. Stark, Massachusetts Institute of Technology, has been appointed Professor at Lamar State College of Technology.

Associate Professor C. W. Topp, Fenn College, has been promoted to Professor.

Assistant Professor H. G. Tucker, University of Oregon, has been appointed Assistant Professor at the University of California, Riverside.

Assistant Professor H. E. Weissler, St. Mary's University, Texas, has a position as an assistant technical director, Falstaff Brewing Corporation, St. Louis, Missouri.

Miss Cecilia T. Welna, University of Massachusetts, has been appointed Instructor at Hillyer College.

Associate Professor J. L. Zemmer, University of Missouri, is on leave of absence and has been appointed a visiting fellow at Yale University.

Mr. H. C. Boardman, Director of Research, Chicago Bridge and Iron Company, Illinois, died on August 6, 1956. He was a member of the Association for thirteen years.

Professor Emeritus W. H. Kirchner, University of Minnesota, died on October 8, 1956. He was a member of the Association for thirty-four years.

Professor G. H. MacCullough, Worcester Polytechnic Institute, died on October 15, 1956. He was a member of the Association for seven years.

Mr. P. A. Piza, San Juan, Puerto Rico, died on November 5, 1956. He was a member of the Association for ten years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTIETH ANNUAL MEETING OF THE ASSOCIATION

The fortieth annual meeting of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on Saturday, December 29, 1956, in conjunction with the annual meetings of the American Mathematical Society and the Association for Symbolic Logic. There were registered 623 persons, including 410 members of the Association.

Sessions of the Association were held on Saturday morning and afternoon in the Upper Strong Auditorium of the University of Rochester. Professor Selby Robinson presided at the morning session and President W. L. Duren presided at the annual business meeting and at the afternoon session. The Program Committee for the meeting consisted of R. V. Churchill, Chairman; M. Gweneth Humphreys, and Selby Robinson.

FIRST SESSION OF THE ASSOCIATION

"Mathematics for the Undergraduate: Some Brave Experiments and Cogent Lessons," by Professor A. E. Ross, University of Notre Dame.

Instruction by Television or Films

"At the University of Washington," by Professor C. B. Allendoerfer, University of Washington.

"At Washington University, St. Louis," by Professor R. R. Middlemiss, Washington University.

"Activities at Other Places," by Professor P. S. Jones, University of Michigan.

"Retiring Presidential Address: Maintaining Communication," by Professor E. J. McShane, University of Virginia.

SECOND SESSION OF THE ASSOCIATION

Annual Business Meeting and Award of the Chauvenet Prize.

"Linear and Quadratic Programming," by Professor A. W. Tucker, Princeton University.

"Prime Numbers," by Professor J. B. Rosser, Cornell University.

"A Unifying Concept in n-dimensional Geometry," by Professor Walter Prenowitz, Brooklyn College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Friday afternoon in the Welles-Brown Room of the Rush Rhees Library, with seventeen members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Duren of the following Nominating Committee for 1957: H. W. Brinkmann, Chairman; R. L. Jeffery, and T. L. Wade.

Professor B. W. Jones of the University of Colorado was elected Second Vice-President for 1957–1958.

The New Jersey Section was officially constituted and Dean A. E. Meder, Jr., of Rutgers University was elected as Governor from the New Jersey Section to serve until June 30, 1958.

The Board voted to accept with an expression of gratitude the following grants: (a) from the National Science Foundation $1,200 for a planning meeting of the Committee on Films for Classroom Instruction; (b) from the National Science Foundation $55,200 for a continuation during 1957–1959 of the Program of Visiting Lecturers; and (c) from the Ford Foundation $150,000 for the continued support of the Committee on the Undergraduate Program in Mathematics.

It was voted to hold the Annual Meeting normally scheduled for December 1958 during the latter part of January 1959.

The Editor was authorized to publish four 96-page issues of the MONTHLY during 1957.

On the recommendation of the Committee to Study the Activities of the Association, the Board voted to request the Policy Committee for Mathematics to arrange as soon as possible a meeting of representatives of all mathematical organizations looking toward the establishment of an American Institute of Mathematics.

On the recommendation of the Committee to Recommend an Association Headquarters and to Nominate a Secretary-Treasurer, the Board voted to re-elect Professor H. M. Gehman for another five-year term (1958–1962) as Secretary-Treasurer of the Association. The Board also directed the Executive and Finance Committees to arrange that the University of Buffalo be compensated for a portion of Professor Gehman's salary so as to permit a corresponding reduction in his academic duties, that he be personally compensated for services during the summer months, that similar financial arrangements be made for an Associate Secretary, and that the Association pay rent for the quarters provided by the University of Buffalo.

It was voted to waive the usual initiation fee in the case of high school students (as well as college undergraduates) provided that the membership application blank is accompanied by the proper certification of status.

The following motion prepared by Professor Saunders MacLane was adopted:

Carl Allendoerfer is now approaching the end of his five year term of service as Editor-in-Chief of the American Mathematical Monthly. His service has been marked by a lively and progressive editorial policy and practice. He has cajoled authors, prodded referees and encouraged associate editors; he has read proof, dictated letters, and met innumerable problems, all to the end of producing a

stimulating journal for the members of the Association. The Board of Governors of the Mathematical Association of America takes this occasion to express by resolution its heartfelt appreciation of these his services. In the spirit of the MONTHLY articles: "What is X?" we can answer the question "What is a good editor?" by "Carl Allendoerfer."

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting was held on Saturday, December 29, 1956 in the Upper Strong Auditorium of the University of Rochester, Rochester, New York. President W. L. Duren presided.

The Secretary announced the results of the balloting for officers in which 1673 votes were cast: Professor G. B. Price of the University of Kansas was elected President for the two-year term 1957–1958, and Professors H. M. Bacon of Stanford University and J. R. Mayor of the University of Wisconsin were elected Governors for the three-year term 1957–1959.

The membership of the Association on December 26 was 6491, a gain of 424 since the beginning of 1956.

The three amendments to the By-Laws printed in the Program of the Rochester Meeting were adopted by the Association.

Reports were made by the chairmen of the following committees: B. W. Jones for the Committee on Visiting Lecturers; E. J. McShane for the Committee on the Undergraduate Program in Mathematics; G. B. Price for the Committee to Study the Activities of the Association; C. B. Allendoerfer for the Committee to Recommend an Association Headquarters and to Nominate a Secretary-Treasurer.

AWARD OF THE CHAUVENET PRIZE

The 1956 Chauvenet Prize was awarded to Professor R. H. Bruck of the University of Wisconsin for his paper entitled "*Recent Advances in the Foundations of Euclidean Plane Geometry*" published in Slaught Paper No. 4 (this MONTHLY, vol. 62 (1955) no. 7, part II, pp. 2–17). This award carries with it a cash prize of $100.

The 1956 Chauvenet Prize is awarded for a noteworthy expository paper published in English during the three-year period 1953–1955 by a member of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. This is the eleventh award of the Chauvenet Prize since its institution by the Association in 1925.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Thursday, December 27, through Saturday. The annual Gibbs Lecture, entitled "Mathematics and the future of science" was given by Professor M. H. Stone of the

University of Chicago. An invited address was delivered by Professor D. C. Spencer of Princeton University.

The Association for Symbolic Logic met on Thursday, December 27, at which time an invited address was given by Dr. G. Kreisel of the Institute for Advanced Study.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Walter Rudin, Chairman; E. H. Batho, Dorothy L. Bernstein, H. M. Gehman, N. G. Gunderson, Hewitt Kenyon, R. W. MacDowell, R. D. Schafer.

Registration headquarters were located in the Women's Residence Hall of the University of Rochester. Dormitory accommodations and meals were also provided in the Women's Residence Hall. Tea was served on Thursday afternoon at the Faculty Club. A conducted tour of the George Eastman House Museum of Photographic Art was held on Friday afternoon. The Employment Register and the Book Exhibit were on display in the basement of the Women's Residence Hall.

A banquet was held on Friday evening at which Professor J. F. Randolph acted as toastmaster. Mr. Sol Linowitz, a trustee of the University of Rochester, welcomed the visiting mathematicians on behalf of the University. Responses were given by President Richard Brauer of the American Mathematical Society, President W. L. Duren of the Mathematical Association of America, and President S. C. Kleene of the Association for Symbolic Logic. Professor M. H. Stone told of the activities of the International Mathematical Union. Professor R. L. Jeffery presented a resolution of thanks to our hosts at the University of Rochester for having planned so well for the comfort and pleasure of the visiting mathematicians.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma, on October 26, 1956. Professor Truman Wester, Chairman of the Section, presided. There were 93 persons in attendance, including 51 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor O. P. Sanders, University of Arkansas; Vice-Chairman, Professor W. A. Rutledge, University

of Tulsa; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma. At the business meeting, the Section voted to participate in the 1958 M.A.A. sponsored High School Mathematics Contest. The group voted to continue meeting in conjunction with the Oklahoma Education Association in the fall and to hold a separate spring meeting for research papers

The following papers were presented:

1. Principal address: *Teacher education and the college entrance examination board commission on mathematics*, by Professor Henry Van Engen; Editor, *Mathematics Teacher*, Iowa State Teachers College.

2. p (mod 30) *and its contents*, by Professor G. E. Meador, Oklahoma City University.

The representation of p(mod 30) is discussed as a special case of p(mod M) where p and M are relatively prime. Thus p(mod 30) equals $30q$ plus p and consists of eight arithmetic sequences whose first terms are; 1, 7, 11, 13, 17, 19, 23, and 29, with 30 the common difference in each case. Another representation of p(mod 30) is indicated by these eight numbers located on a circle which has been divided into thirty units. The extent to which circular addition and circular multiplication may be applied is discussed. Finally p(mod 30) is shown to contain all primes except 2, 3, and 5, and every product of two or more of these numbers is a composite number belonging to the group.

3. *The supplementary training program for high school science teachers*, by Professor J. H. Zant, Oklahoma Agricultural and Mechanical College.

The purpose of this supported program is to up-grade science and mathematics teaching by requiring these 50 high school teachers to take a 9-months program of graduate courses in the fields of biology, chemistry, mathematics, physics and engineering. It is hoped by this means to enable them to inspire and motivate more students to go to college and study science and engineering. Special courses, designed to accomplish the above purpose, have been organized. All are courses in science and are taught by productive, research scientists from the College staff, augmented by scientists from other institutions. Such programs will be continued next year and the number of institutions sponsoring them will be increased to approximately fifteen.

4. *Certain rationality relationships in triangles*, by Professor H. W. Linscheid, Southwestern State College.

Rationality of the three sides of a triangle and of its circumdiameter is a sufficient condition for the rationality of the trigonometric functions of the angles of the triangle, the length of the inradius, the lengths of the exradii, and the area of the triangle. A parametric representation is given for each of these quantities. A parametric representation is given also for the sides of a triangle containing an angle of 60° or 120° and for which the sides are represented by rational numbers.

5. Luncheon Address: *Modernizing the secondary school mathematics curriculum*, by Professor Henry Van Engen, Iowa State Teachers College.

6. Discussion: *What to do about college freshmen not prepared for mathematics courses on the college level*, led by Professor L. W. Johnson, Oklahoma Agricultural and Mechanical College.

The discussion leader introduced the proposals:

A. That Oklahoma colleges cease teaching high school courses such as solid geometry, elementary and intermediate algebra and similar remedial courses during the regular term and that remedial mathematics be taught only during the summer term when college plants and staffs are not overtaxed.

It was pointed out that this would provide better utilization of both professional staff and

space and that it would operate to improve the economic status of both college and high school mathematics teachers by expanding the opportunity for 12-months employment.

It was argued that the inconvenience to both parents and graduating high school seniors would operate to return the problem to the high school, where it properly belongs; also, that, "The shock to parents in learning that their children are not adequately prepared for college should occur while the children are in the local community, not later when they are away in college." It is hoped that the policy might eventually cause high schools to provide their own remedial instruction.

B. That the major colleges band together in this action, and in devising an examination to be given in the high schools during either the junior or senior year to let students who are so weak they must take remedial mathematics know about it in time to make proper preparation before attempting to enter a college or university in September.

It was recognized that some sort of High School "Exit" Examinations would have to be inaugurated to determine who would be required to attend college the summer before entering college in the fall. The feasibility of such examinations was discussed.

C. That high schools be encouraged to offer remedial courses to 12th grade students desiring them before they leave high school.

The discussion was lively. There was not complete agreement that the proposals were workable, but the consensus of opinion was that they had much merit and should be given serious consideration.

R. V. ANDREE, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The organizational meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers University, New Brunswick on November 3, 1956. Dean A. E. Meder, Jr., of Rutgers University presided over the morning session. Professor D. R. Davis of the New Jersey State Teachers College at Montclair presided over the afternoon session. In the unavoidable absence of the Secretary-Treasurer minutes for the latter part of the afternoon session were recorded by Mr. R. S. Lockhart, Madison High School, Madison. There were 71 persons present including 61 members of the Association.

At the business meeting By-Laws prepared by a committee under the chairmanship of Professor C. A. Nelson of Douglass College, Rutgers University, were adopted, subject to minor changes to be made by the Executive Committee. The following officers were elected: Chairman, Dean A. E. Meder, Jr., Rutgers University; Secretary-Treasurer, Professor I. L. Battin, Drew University; Members at Large of the Executive Committee: for one year, (Senior Member, Chairman of the Program Committee), Professor D. R. Davis, New Jersey State Teachers College at Montclair; for two years, Professor S. S. Wilks, Princeton University; for three years, Dr. H. O. Pollak of the Mathematical Research Department, Bell Telephone Company Laboratories, Murray Hill. The Executive Committee was authorized to make arrangements for another annual meeting in the fall of 1957. The Executive Committee was also authorized to investigate the possibility of holding a joint meeting with the Association of Mathematics Teachers of New Jersey, the time, possibly, to be in the spring.

Professor D. R. Davis served as Chairman of the ad hoc Program Committee. The following papers were presented:

1. *Mathematics in communication*, by Dr. Brockway McMillan, Assistant Director of Systems Engineering, Bell Telephone Laboratories, Murray Hill. (By invitation).

The following diagram illustrates the important role which mathematics has played in the

past, and is now playing, in the field of communication. Statistical generalization is required in passing from *A* to *B*. Statistics are important in all topics.

	CIRCUITS	TRAFFIC
TRANSMISSION	*A* Linear Graphs Differential Equations LaPlace Transform Functions of a Complex Variable	*B* Random Processes Fourier Integral
SWITCHING	*C* Linear Graphs Boolean Algebra Combinatory Analysis	*D* Random Walk (Markov Processes)

2. *Mathematics for digital computers*, by Mr. G. W. Kays, International Business Machines Corporation. (By invitation).

Electronic computers answer the need of the scientist and industrialist for extensive and rapid calculations. The high school student who continues his studies in mathematics will find many opportunities for a career in this rapidly expanding field. Aeronautics, market analysis, and production control are only a few of the fields that use this powerful tool. Examples of the type of mathematics applications shown included the extraction of a square root, the roots of a polynomial by synthetic division, the simultaneous solution of linear equations with elements of linear programming. The philosophies and techniques of elementary mathematics as they apply to the computer field were emphasized.

3. *Boolean algebra and its role in switching theory*, by Professor A. H. Diamond, Stevens Institute of Technology. (By invitation).

A Boolean algebra of two elements can be applied to the design of relay networks by interpreting the expressions of the algebra appropriately, *e.g.* "$a+bc'$" is interpreted to mean "relay a is closed or relay b is closed and relay c is open." A network of given design determines a formula which has the value 1 or 0, when certain relays are closed and others open, according as the circuit as a whole is closed or open. A fundamental problem is that of determining formulas corresponding to a minimum number of relay contacts for a given design.

I. L. BATTIN, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The second annual meeting of the Northeastern Section of the Mathematical Association of America was held at the University of Connecticut in Storrs, Connecticut, on November 24, 1956. Professor Howard Eves, Chairman of the Section, presided at the morning session and Professor Stanley Bezuszka, S.J., Vice-Chairman of the Section, presided at the afternoon session. Seventy-one persons attended the meeting. Of these, sixty were members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Stanley Bezuszka, S.J., Boston College; Vice-Chairman, Professor D. E. Richmond, Williams College; Secretary-Treasurer, Professor R. E. Johnson, Smith College. It was voted that the Chairman appoint a committee to study the question of sponsoring the High School Contest within the territory of the Section, said committee to present a recommendation at the next annual meeting of the Section.

The following papers were presented by invitation:

1. *An integral transform related to heat conduction*, by Professor D. V. Widder, Harvard University.

A temperature function is a solution $u(x, t)$ of the partial differential equation $\delta^2u/\delta x^2=\delta u/\delta t$. The source solution is $k(x, t)=(4\pi t)^{-1/2}\exp(-x^2/4t)$ for $t>0$. The chief purpose of the present study is to investigate the transform $f(t)=\int_{-\infty}^{\infty}k(y, t)\phi(y)dy$. If this integral converges absolutely for $t>0$, we say that $f(t)\in A$. If $D^{-1/2}f(t)\in A$, where the Riemann-Liouville definition for the fractional integral $D^{-1/2}$ is intended, we say that $f(t)\in B$. A typical conclusion is that a temperature function has an absolutely convergent Poisson integral representation $u(x, t)=\int_{-\infty}^{\infty}k(x-y, t)\phi(y)dy$ if and only if $u(0, t)\in A$ and $u_x(0, t)\in B$.

2. *A property of the binomial coefficients*, by Professor F. W. Perkins, Dartmouth College.

This paper contains a discussion of the following theorem, based on a classical result due to Legendre and more recent work of Dickson: Let $n_sn_{s-1}\cdots n_1n_0$ be the representation of a non-negative integer n in a number system with a positive prime p as base. Then the number of those terms in the binomial expansion of $(x_0+x_1)^n$ whose coefficients are not divisible by p is $\prod_{k=0}^{s}(1+n_k)$. This may be generalized for the multinomial expansion of $(x_0+x_1+\cdots+x_m)^n$, where the number of coefficients not divisible by the prime p is $\prod_{k=0}^{s}C_m^{m+n_k}k$. A number of corollaries are discussed.

3. *The Cauchy integral theorem and the Poincaré-Bendixson theory*, by Professor F. M. Stewart, Brown University.

One proof of the Cauchy integral theorem begins by showing that a function analytic in a convex region has an indefinite integral. The invariance of the integral under deformations of the path of integration is then established by an easy direct computation. The same methods can be used to study the index of a curve with respect to a vector field. This simplifies the use of the index in proving results about the integral curves of $x'=f_1(x, y)$, $y'=f_2(x, y)$, while avoiding the use of the Jordan curve theorem in places where it is neither needed nor wholly adequate.

4. *Some algebraic aspects of elementary dimensional analysis*, by Professor D. E. Christie, Bowdoin College.

The symbolism of elementary dimensional analysis is considered as a free group on a finite but arbitrary number of generators. When a large number of generators is used, it is shown that ordinary empirical results induce endomorphisms of the group. An extended system distinguishing vectors from scalars is given a simple matric representation. Such examples are proposed as a means of motivating the teaching of algebra to undergraduate physics students.

5. *Euclides somewhat vindicatus*, by Professor I. H. Rose, University of Massachusetts.

A first approximation to a set of axioms for plane Euclidean geometry, exhibiting certain desirable pedagogical and logical properties, is presented. The axioms stem mainly from work of Fréchet and Hilbert, but embody also several much-maligned ideas of Euclid.

6. *Probability studies in the seventeenth century*, by Professor Oystein Ore, Yale University.

The point of departure is the well-known discussion of probability problems in 1654 between Blaise Pascal and Antoine Gombaud, sieur de Baussay, often grossly misrepresented as the "gambler" de Méré. The personal relations between the two men as well as the historical nature of their problems is elucidated in connection with the history of the development of the probability theory in the 17th century.

R. E. JOHNSON, *Secretary*

OFFICERS AND COMMITTEES AS OF JANUARY 1, 1957

OFFICERS

President, G. B. PRICE, University of Kansas (1957–1958)
First Vice-President, R. V. CHURCHILL, University of Michigan (1956–1957)
Second Vice-President, B. W. JONES, University of Colorado (1957–1958)
Editor, R. D. JAMES, University of British Columbia (1957–1961)
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo (1953–1957)
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo (1953–1957)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

SAUNDERS MACLANE, University of Chicago (1953–1958)
E. J. MCSHANE, University of Virginia (1955–1960)
W. L. DUREN, JR., University of Virginia (1957–1962)

Governors at Large

H. W. BRINKMANN, Swarthmore College (1955–1957)
M. A. ZORN, Indiana University (1955–1957)
A. S. HOUSEHOLDER, Oak Ridge National Laboratory (1956–1958)
M. F. SMILEY, State University of Iowa (1956–1958)
H. M. BACON, Stanford University (1957–1959)
J. R. MAYOR, University of Wisconsin (1957–1959)

Sectional Governors (July 1, 1954–June 30, 1957)

Allegheny Mountain, MORRIS OSTROFSKY, Westinghouse Electric Corporation
Indiana, P. D. EDWARDS, Ball State Teachers College
Kentucky, H. H. DOWNING, University of Kentucky
Metropolitan New York, R. M. FOSTER, Polytechnic Institute of Brooklyn
Nebraska, M. A. BASOCO, University of Nebraska
Northern California, W. H. MYERS, San Jose State College
Oklahoma, L. W. JOHNSON, Oklahoma Agricultural and Mechanical College
Rocky Mountain, C. A. HUTCHINSON, University of Colorado
Wisconsin, R. C. HUFFER, Beloit College

Sectional Governors (July 1, 1955–June 30, 1958)

Kansas, C. B. READ, University of Wichita
Missouri, F. F. HELTON, Central College
New Jersey, A. E. MEDER, JR., Rutgers University
Northeastern, G. B. THOMAS, JR., Massachusetts Institute of Technology
Ohio, L. L. LOWENSTEIN, Kent State University
Pacific Northwest, IVAN NIVEN, University of Oregon
Southeastern, F. W. KOKOMOOR, University of Florida
Southwestern, M. S. HENDRICKSON, University of New Mexico
Upper New York State, J. F. RANDOLPH, University of Rochester

Sectional Governors (July 1, 1956–June 30, 1959)

Illinois, E. C. KIEFER, Millikin University
Iowa, BERNARD VINOGRADE, Iowa State College
Louisiana-Mississippi, Z. L. LOFLIN, Southwestern Louisiana Institute

Maryland-Dist. of Col.-Virginia, O. J. RAMLER, Catholic University of America
Michigan, B. M. STEWART, Michigan State University
Minnesota, G. K. KALISCH, University of Minnesota
Philadelphia, N. J. FINE, University of Pennsylvania
Southern California, P. H. DAUS, University of California at Los Angeles
Texas, C. R. SHERER, Texas Christian University

STANDING COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1956–1959), J. F. RANDOLPH (1954–1957), H. M. GEHMAN, *ex officio*.

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

TIBOR RADO, *Chairman* (1954–1959), P. R. HALMOS (1952–1957), SAMUEL EILENBERG (1953–1958), E. E. FLOYD (1956–1961), W. R. SCOTT (1957–1959).

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

MARK KAC, *Chairman* (1955–1957), S. C. KLEENE (1956–1958), R. P. DILWORTH (1957–1959).

JOINT COMMITTEE ON PLACES OF MEETINGS

C. B. MORREY, JR., *Chairman* (1955–1957), R. M. THRALL (1956–1958), R. D. SCHAFER (1957–1959).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

L. M. KELLY, *Chairman* (July 1954–June 1957), R. E. GREENWOOD (July 1956–June 1959), LEO MOSER (July 1955–June 1958); L. E. BUSH, *Director* (July 1953–June 1958).

COMMITTEE ON SECTIONS

C. H. FRICK (1955–1958), ROY DUBISCH (1957–1960), EDITH R. SCHNECKENBURGER, *ex officio*.

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

R. D. SCHAFER, *Chairman* (1956–1958), H. S. M. COXETER (1955–1957), F. B. WRIGHT (1957–1959), R. D. JAMES, *ex officio*.

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

J. S. FRAME, *Chairman*, H. M. BACON, RAY BERKOWITZ, W. M. HIRSCH, MORRIS OSTROFSKY, G. W. PATTERSON, J. A. WARD.

COMMITTEE ON FILMS FOR CLASSROOM INSTRUCTION

P. S. JONES, *Chairman*, H. H. CAMPAIGNE, MARGUERITE LEHR.

COMMITTEE ON HIGH SCHOOL CONTESTS

W. H. FAGERSTROM, *Chairman*. (Other members to be appointed later).

COMMITTEE TO PREPARE A THIRD EDITION OF "PROFESSIONAL OPPORTUNITIES IN MATHEMATICS"

F. J. WEYL, *Chairman*, L. W. COHEN, A. S. HOUSEHOLDER, C. C. HURD, C. R. SHERER, T. H. SOUTHARD.

COMMITTEE TO STUDY THE ACTIVITIES OF THE ASSOCIATION

G. B. PRICE, *Chairman*, C. W. CURTIS, DAVID GALE, ROTHWELL STEPHENS, R. R. STOLL, PATRICK SUPPES, G. B. THOMAS, JR.

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

E. J. McSHANE, *Chairman*, W. L. DUREN, JR., J. G. KEMENY, G. B. PRICE, A. L. PUTNAM, A. W. TUCKER, R. L. DAVIS, *Executive Secretary*.

COMMITTEE ON VISITING LECTURERS

B. W. JONES, *Chairman*, G. B. HUFF, D. E. RICHMOND, R. A. ROSENBAUM.

NOMINATING COMMITTEE FOR 1957

H. W. BRINKMANN, *Chairman*, T. L. WADE, R. L. JEFFERY.

REPRESENTATIVES OF THE ASSOCIATION

On the Policy Committee for Mathematics:
D. E. RICHMOND (1955–1957), G. B. PRICE, *ex officio*, H. M. GEHMAN, *ex officio*.
On the National Research Council:
H. M. GEHMAN (July 1, 1956–June 30, 1959).
On the Council of the American Association for the Advancement of Science:
H. J. ETTLINGER (1956–1957), M. R. HESTENES (1957–1958).
On the American Council on Education:
G. B. PRICE, *ex officio*, H. M. GEHMAN, *ex officio*.
On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
P. S. JONES (1957–1959).
On the Committee on Definitions of Electrical Terms:
S. A. SCHELKUNOFF.
On a Committee of the N.C.T.M. on a Vocational Pamphlet:
A. L. PUTNAM.
On the Committee on Mathematical Training of Social Scientists:
A. W. TUCKER.

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION AS OF JANUARY 1, 1957

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1950. For information about preceding years, consult the American Mathematical Monthly for March 1955.)

PRESIDENT

E. R. HEDRICK	1916	E. T. BELL	1931–1932
FLORIAN CAJORI	1917	ARNOLD DRESDEN	1933–1934
E. V. HUNTINGTON	1918	D. R. CURTISS	1935–1936
H. E. SLAUGHT	1919	A. J. KEMPNER	1937–1938
D. E. SMITH	1920	W. B. CARVER	1939–1940
G. A. MILLER	1921	R. W. BRINK	1941–1942
R. C. ARCHIBALD	1922	W. D. CAIRNS	1943–1944
R. D. CARMICHAEL	1923	C. C. MACDUFFEE	1945–1946
H. L. RIETZ	1924	L. R. FORD	1947–1948
J. L. COOLIDGE	1925	R. E. LANGER	1949–1950
DUNHAM JACKSON	1926	SAUNDERS MACLANE	1951–1952
W. B. FORD	1927–1928	E. J. MCSHANE	1953–1954
J. W. YOUNG	1929–1930	W. L. DUREN, JR.	1955–1956

Vice-President

N. H. McCoy	1949–1950	W. L. Duren, Jr.	1953–1954
L. M. Graves	1950–1951	H. S. M. Coxeter	1954–1955
Jewell H. Bushey	1951–1952	G. B. Price	1955–1956
F. L. Griffin	1952–1953		

Secretary-Treasurer

W. D. Cairns	1916–1942	W. B. Carver	1943–1947

Editor

C. V. Newsom	1947–1951	C. B. Allendoerfer	1952–1956

Governor (arranged alphabetically)

C. R. Adams	1948–1950	Ralph Hull	1946–1948, 1954–1956
E. B. Allen	1949–1952	S. B. Jackson	1953–1956
C. B. Allendoerfer	1949–1951	C. G. Jaeger	1950–1953
W. L. Ayres	1948–1950	R. D. James	1952–1955
R. H. Bardell	1948–1951	B. W. Jones	1951–1953
I. A. Barnett	1952–1955	P. S. Jones	1953–1956
C. F. Barr	1951–1954	M. S. Knebelman	1949–1952
T. A. Bickerstaff	1950–1953	D. H. Lehmer	1947–1949, 1952–1954
J. W. Bradshaw	1947–1950	A. J. Lewis	1948–1951
H. E. Bray	1947–1950	F. A. Lewis	1952–1955
J. C. Brixey	1951–1954	K. O. May	1953–1956
B. H. Brown	1952–1955	R. J. Michel	1952–1955
M. C. Brown	1948–1951	E. B. Miller	1953–1956
L. E. Bush	1947–1950	F. H. Miller	1948–1951
S. S. Cairns	1953–1955	W. E. Milne	1952–1954
R. H. Cameron	1950–1953	F. R. Morris	1948–1951
C. C. Camp	1948–1951	C. W. Munshower	1952–1955
G. R. Clements	1950–1953	C. O. Oakley	1953–1956
T. F. Cope	1951–1954	E. N. Oberg	1950–1951, 1954–1956
N. A. Court	1945–1947, 1948–1951	H. P. Pettit	1951–1954
H. S. M. Coxeter	1945–1947, 1951–1953	J. C. Polley	1951–1954
W. M. Davis	1951–1953	G. B. Price	1952–1955
H. L. Dorwart	1948–1951	G. E. Raynor	1950–1953
W. L. Duren, Jr.	1947–1950	F. A. Rickey	1953–1956
J. M. Earl	1951–1954	E. B. Roessler	1951–1954
G. M. Ewing	1949–1952	J. B. Rosenbach	1951
Tomlinson Fort	1949–1952	R. G. Sanger	1949–1952
J. S. Frame	1950–1953	I. S. Sokolnikoff	1947–1950
Philip Franklin	1954–1956	C. E. Springer	1954–1956
A. E. Gault	1950–1953	E. P. Starke	1947–1950
R. E. Gilman	1949–1952	F. H. Steen	1951–1954
R. F. Graesser	1952–1955	H. P. Thielman	1947–1950
D. W. Hall	1947–1950	C. W. Trigg	1953–1956
E. H. Hanson	1950–1953	A. W. Tucker	1953–1955
E. R. Heineman	1953–1956	Earl Walden	1949–1952
M. R. Hestenes	1950–1952	R. J. Walker	1949–1951
E. H. C. Hildebrandt	1947–1950	Marie J. Weiss	1950–1952
D. L. Holl	1953–1954	F. B. Wiley	1949–1952
Aughtum S. Howard	1951–1954		

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to January 1, 1957)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission, endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee, advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers

of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted. The by-laws of each Section when organized and any subsequent changes in these by-laws must be approved by the Board. The Board shall maintain general supervision over the activities of all Sections.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be five dollars ($5), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of two dollars ($2).

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (⅔) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS, University of Kansas, Lawrence, April 13, 1957.

KENTUCKY, Berea College, Berea, April 27, 1957.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.

METROPOLITAN NEW YORK, Hunter College, New York, April 27, 1957.

MICHIGAN, Wayne State University, Detroit, March 23, 1957.

MINNESOTA, Carleton College, Northfield, May 11, 1957.

MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.

NEW JERSEY, Fall, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 28, 1957.

NORTHERN CALIFORNIA

OHIO, University of Cincinnati, April 20, 1957.

OKLAHOMA, University of Arkansas, Fayetteville, April 12–13, 1957.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.

PHILADELPHIA, November 28, 1957.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

THE HERBERT ELLSWORTH SLAUGHT MEMORIAL PAPERS

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. The following five numbers have already been published:

1. *Fourier Series; The Genesis and Evolution of a Theory* by R. E. Langer, v+86 pages.
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3. *Proceedings of the Symposium on Special Topics in Applied Mathematics*. iv+73 pages.
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